SECTIONS FOR SEMIFLOWS AND KAKUTANI SHIFT EQUIVALENCE

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Abstract. We show here that the basic theory of sections for flows preserving a probability measure and Kakutani equivalence for measure preserving endomorphisms can be lifted to the case of semiflows (actions of $\mathbb{R}^+$) and endomorphisms if one replaces conjugacy by shift equivalence. This leads to a notion of "Kakutani shift equivalence", that two endomorphisms induce shift equivalent actions. We give a variety of characterizations of this property. We develop a theory of time changes for semiflows and indicate a variety of directions to pursue, both for semiflows and for shift equivalence of endomorphisms.

1. Introduction

Our goal is to take the classical theory of time-changes and sections for measure preserving $\mathbb{R}$ actions (flows) and of Kakutani Equivalence [6] [13] and to show the natural form they take when extended to the study of measure preserving $\mathbb{R}^+$ actions (semiflows). Krengel [11] has shown that one can lift the Ambrose-Kakutani construction of cross sections for measure preserving flows to the case of semiflows. He in fact proves the existence of sections for recurrent and nonsingular semiflows, not just finite measure preserving semiflows. His construction of sections follows that of Ambrose and Kakutani [1]. We will repeat this basic construction of cross-sections here for free and measurable actions of $\mathbb{R}^+$ preserving a probability measure. That is to say we will construct sections as standard probability spaces in their own right with measure preserving return maps. Ambrose and Kakutani and Krengel do not require ergodicity of the action and neither will we. We include this well-known construction as it is brief and provides necessary perspective for the rest of your work. Beyond the construction of the section and return map though we will need to require the return-time to the section to be bounded away from zero in order to insure the section is a finite measure space. The easiest way to get this is to assume ergodicity at this point and so we will.

Key words and phrases. hi.

Whenever an "action" of a semigroup is discussed we will mean a free, measure preserving and measurable action of the semigroup of transformations acting on a standard probability space. The semigroups we will consider will never be more than $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{Z}^+$ and $\mathbb{Z}$ so we make no further general assumptions. Our standard probability spaces will be written as $(X, \mathcal{F}, \mu)$ or $(Y, \mathcal{G}, \nu)$. We will often abbreviate these as just X or Y as the context will make the algebra of measurable sets and measure clear. The same assumption of measurability, measure preservation and freeness will apply whenever we use the word "flow", "semiflow", "automorphism", or "endomorphism" on a measure space.

In order to understand the flow-under-a-function picture for semiflows better and the notion of Kakutani equivalence of return maps for semiflows we will introduce another idea.

Definition 1.1. *We say that two semigroup actions* $\{T_g\}_{g\in G}$ *and* $\{S_g\}_{g\in G}$ *acting on* X *and* Y *are "shift-equivalent" if there are measure preserving factor maps* $U\colon X \to Y$ *and* $V: Y \to X$ *with*

$$S_g U = U T_g$$
$$T_g V = V S_g \text{ and}$$
$$VU(x) = T_{g(x)}(x) \text{ is on the same orbit as } x.$$

The value $g(x)$ is called the "lag" at x of the shift equivalence given by U, V. One calculates that $UV(y) = UVU(x) = U(T_{g(x)}(x)) = S_{g(x)}(y)$. Moreover if G is abelian then it is easily seen that $\ell(T_g(x)) = g(x)$. That is to say the lag is a constant across an orbit.

The notion of shift equivalence was first introduced by Williams for his classification of generalized solenoid maps [17]. In the context of the Williams conjecture it has played a significant role in classification theory of shifts of finite type [18].

In the case of group actions, shift equivalence implies conjugacy. In fact in this case both U and V will be conjugacies. This is perhaps why this notion has not played any significant role in measurable dynamics to date. We will see that when attempting to apply the classical arguments of the theory of Kakutani equivalence for flows to the case of semiflows it is natural to replace conjugacy by shift equivalence at various points.

Krengel, in [11] offers a generalization to semiflows of the flow-under-a-function-picture. Once one has constructed a cross-section X_0 and return time function f to that section, one can construct the flow-under-a-function as an action of $\mathbb{R}^+$ on the subset of $X_0 \times \mathbb{R}^+$ that lies under the graph of f. For semiflows though this new action will only factor onto the original flow and will not necessarily be conjugate to it. Krengel's approach in [11] is to pull back to this flow-under-a -unction space the σ-algebra of measurable sets and characterize what it must look like. We will not follow this path here.

Rather, we will replace conjugacy by shift equivalence. We will see that so long as the return time function is bounded, the original semiflow and the flow under a function semiflow will be shift equivalent. We will show something a bit more general. Suppose we have two semiflows and are given cross-sections for them. Suppose the return maps to these sections are shift equivalent by maps $\bar{U}, \bar{V}$. Moreover, suppose the return-time functions are bounded and agree at $\bar{x}$ and $\bar{V}(\bar{x})$. We will see that this implies the two semiflows are shift-equivalent. Conversely, if we have two shift-equivalent semi-flows, then any section of one will be shift equivalent to some section of the other where the return-time functions agree at $\bar{x}$ and $\bar{V}(\bar{x})$.

We will next consider time changes of semiflows and show how they may be represented by varying the return time to a section. This generalizes to the case of semiflows the work of Kubo in [12]. This mirrors for semiflows the behavior for flows and is not difficult but is worthy of mention. What is of importance here is that the existence of sections for semiflows means one has an easy approach to constructing and smoothing time changes, in complete analogy to the case of flows.

Our next step is to consider the relation among return maps to sections of some fixed semiflow. The classical theory of Kakutani equivalence [6] begins with the fact that two measure-preserving transformations arise as return maps to sections of some common flow iff they induce conjugate actions on subsets. This relation, of inducing conjugate actions, is the classical notion of Kakutani equivalence. It is important to note here that this fact arises naturally in that the conjugacy of the induced maps will pair points which are found to lie on the same flow orbit when the two actions are realized on sections of a common flow.

Definition 1.2. *We say two that measure-preserving and ergodic endomorphisms T and S acting on probability spaces X and Y are "Kakutani shift-equivalent" if there are subsets $A \subseteq X$ and $B \subseteq Y$ of positive measure where T_A and S_B are shift equivalent.*

We will see that this is an equivalence relation. In the case of semiflows we will see that the return maps to distinct sections induce shift-equivalent actions on subsets. See Theorem 5.3 for a complete list of conditions equivalent to Kakutani shift equivalence. If one wishes to maintain the naturalness of the shift-equivalence (that corresponding points for the factor maps should lie on the same flow orbit in the common flow) then this is the best one can expect. We will not obtain the converse, that Kakutani shift-equivalent maps arise as sections of a common flow. Rather they will arise as sections of shift-equivalent flows and that symmetrically they are shift equivalent to actions that arise as return maps to sections of a common flow. These still fulfill the general theme of our work here.

Definition 1.3. *A pair of factor maps U, V give an "elementary shift-equivalence" between two endomorphisms T and S if they give a shift equivalence of lag 1, i.e. $VU = T$ and $UV = S$.*

We will see that if T and S are Kakutani shift-equivalent then they induce maps that have an elementary equivalence. We will also see that when two endomorphisms have an elementary shift equivalence then they do indeed arise as return maps to sections of a common flow.

2. Constructing Cross-Sections

Take T_t, $t \in \mathbb{R}^+$, to be a measurable and measure-preserving semiflow on the standard probability space $(X, \mathcal{F}, \mu)$. As a first and useful step we want to see that T_t has a cover on which the action is invertible. For $\mathbb{N}$-actions this is the standard inverse limit construction. As we want the cover to be a standard space we don't want to blindly pursue the inverse limit. We will assume the reader is familiar with the inverse-limit construction for $\mathbb{N}$-actions. It is extremely simple to extend this to flows and for completeness we include the argument.

Theorem 2.1. *Suppose T_t, $t \in \mathbb{R}^+$, is a measurable and measure-preserving semiflow on the standard probability space $(X, \mathcal{F}, \mu)$. There is then a measurable and measure-preserving flow (action of $\mathbb{R}$) $\hat{T}_t$, on a standard probability space $(\hat{X}, \hat{\mathcal{F}}, \hat{\mu})$ that factors onto T_t. It is the unique minimal such cover in that any other such invertible cover must factor through $\hat{X}$.*

Proof. Take $(\hat{X}, \hat{\mathcal{F}}, \hat{\mu})$ to be the inverse limit of the endomorphism T_1. That is to say

$$\hat{X} = \{\{x_i\}_{i\in\mathbb{Z}} \mid T_1(x_i) = x_{i+1}\}.$$

The algebra $\mathcal{F}$ and measure $\hat{\mu}$ lift naturally to $\hat{X}$ and the map $\{x_i\} \to \{T_1(x_i)\}$ lifts the action of T_1 to this cover as the shift map on sequences. It is easily seen and standard that this cover is the unique minimal invertible cover of T_1 in that any other such must factor through this one. Notice that any S that is measure preserving and commutes with T_1 will lift to an $\hat{S}$ on $\hat{X}$ by $\hat{S}(\{x_i\}) = \{S(x_i)\}$. Hence all maps T_t lift to maps $\hat{T}_t$ that commute with $\hat{T}_1$. Notice the integer times remain well-defined here. Together $\hat{T}_t$, $t \in \mathbb{R}^+$, and $\hat{T}_n$, $n \in \mathbb{Z}$, generate an action of $\mathbb{R}$ giving the lift we seek. Any other lift must factor through this one as the time 1 action must. One checks that the intermediary points must also project through the $\hat{X}$ lift correctly. $\square$

We now present show that the Ambrose-Kakutani construction is in fact "one-sided", i.e. holds for the semiflow T_t. As indicated earlier, this a special case of the work of Krengel in [10], who showed that not only does it apply to endomorphic actions but invariance of the measure is not needed, just non-singularity and recurrence. We will modify slightly the argument of Ambrose and Kakutani to make our work somewhat more transparent. Our approach is similar but not identical to that of Krengel in [10]. These variations are perhaps just a matter of taste, the core idea is always the same.

Definition 2.1. *Suppose T_t is a semiflow on the probability space $(X, \mathcal{F}, \mu)$. A "thick section" for the flow is a set $F \in \mathcal{F}$ along with parameters $0 < \alpha < \beta$ so that there is a measurable function $\beta(x) > 0$ and moreover $\beta(T_{\beta(x)}(x)) \geq \beta$ so that*

$$\{T_t(x)\}_{\beta(x) \leq t < \alpha + \beta(x)} \subseteq F \text{ and } \{T_t(x)\}_{\alpha+\beta(x) \leq t < \alpha + \beta(x) + \beta(T_{\beta(x)}(x))} \cap F = \emptyset.$$

What this says is that along any forward orbit of the semiflow we will see countably infinitely many subintervals of the time parameter the form $I_i(x) = [t_i, t_i + \alpha)$ where $T_t(x) \in F$ for $t \in I_i$. Two of these intervals must be separated by an interval of time of length $> \beta - \alpha$ where the points are not in F. The value $\beta(x)$ is the time to the next beginning of an interval in F. The value α is the "thickness" of the section. Notice that if we have a section of thickness α it is easy to intersect it with some collection of rational inverse images $T_{-q}(F)$ to form a section of any smaller thickness whose intervals have the same left endpoints.

We construct thick sections as follows:

Theorem 2.2. *For a measurable, measure preserving and free semiflow T_t on a standard probability space $(X, \mathcal{F}, \mu)$, X can be written as an at most countable union of disjoint measurable and invariant sets X_i on each of which we have a thick section F_i for some parameters $0 < \alpha_i < \beta_i$. If T_t is ergodic then there is only a single set and T_t has thick sections.*

Proof. First select a set $A \subseteq F$ that intersects all ergodic components of T_t in subsets of intermediate measure. As T_t acts freely, the Rokhlin decomposition of $\mathcal{F}$ over the algebra of T_t-invariant sets has nonatomic fiber measures. Hence we can in fact take A to have fiber measure 1/2 for all ergodic measures. We note here that this argument was unavailable to Ambrose and Kakutani in 1941. They worked a bit to obtain such as subset.

For each value $c > 0$ one now constructs the function

$$\phi_c(x) = \frac{1}{c}\int_0^c \chi_A(T_t(x))\,dt.$$

Measurability of T_t guarantees this is an $\mathcal{F}$-measurable function. Moreover for a.e. x,

$$\lim_{c\to 0}\phi_c(x) = \chi_A(x)$$

and for each c, $\phi_c(T_t(x))$ is Lipschitz continuous in t. In fact

$$|\phi_c(T_t(x)) - \phi_c(T_s(x))| \le \frac{2|t-s|}{c}.$$

Set

$$A_c = \{x \mid \phi_c(x) > 3/4\} \quad \text{and}$$
$$B_c = \{x \mid \phi_c(x) < 1/4\}.$$

Now let

$$G_c = \{x \mid T_t(x) \in A_c \,\text{and}\, T_s \in B_c \,\text{for values}\, t \nearrow \infty, s \nearrow \infty\}.$$

The sets G_c are invariant sets that exhaust X as $c \searrow 0$. Hence we can write X as a countable disjoint union of T_t-invariant sets X_i each of which is contained is some G_{c_i}.

Let

$$F_i = \{x \mid \text{for some } 0 < t_0 \le c_i/16, \phi_{c_i}(T_{t_0}(x)) = 1/2$$
$$\text{and for all } 0 < t < c/8,\ \phi_{c_i}(T_{t_0+t}(x)) < 1/2\}.$$

One checks that:

(1) F_i is $\mathcal{F}$ measureable. Relying on the continuity in t of $\phi_{c_i}(T_t(x))$ it can be described using countably many boolean operations on ϕ_{c+i}-measurable sets.
(2) Using that $\phi_{c_i}(T_t(x))$ is actually Lipschitz in t, between an occurence of A_{c_i} and B_{c_i} on any orbit there must be at least one occurrence of F_i.
(3) Occurrences of F_i on an orbit come in time intervals of the form $[t, t + \frac{c_i}{16})$ seperated by gaps at least $c_i/16$ long.

Hence F_i is a thick section with parameters $\alpha = c_i/16$ and $\beta = c_i/8$. □

Definition 2.2. *A set $\bar{F}$ is a "section" for a semiflow T_t if for some thick section F, $\bar{F}$ consists of all those x at the left end-points of intervals of occurrence of F on T_t-orbits. We make $\bar{F}$ a measure space as follows. A subset $\bar{A} \subseteq \bar{F}$ is measurable if its thickening given by*

$A = \{T_t(x) \mid x \in F, 0 \le x < \alpha\}$ is $\mathcal{F}$ measurable. It is easy to check this σ-algebra of sets, we call $\bar{\mathcal{F}}$, separates points a.s. on $\bar{F}$. We put a measure on $\bar{\mathcal{F}}$ by setting $\bar{\mu}(\bar{A}) = \mu(A)/\mu(F)$.

The definition of $\bar{\mathcal{F}}$ and $\bar{\mu}$ are easily seen to not depend on the choice of thick section over $\bar{F}$. The thick section over $\bar{F}$ of thickness α we call the "thickening of $\bar{F}$ by α.

Definition 2.3. *For $\bar{F}$ a section of T_t acting on X, define $r_{\bar{F}}(\bar{x}) = \inf\{t \in \mathbb{R}^+ \mid T_t(\bar{x}) \in \bar{F}\}$. This is the "return time" to the section $\bar{F}$. Set $\bar{T}(\bar{x}) = T_{r(\bar{x})}(\bar{x})$, the "return map"to the section.*

Corollary 2.1. *For $\bar{F}$ a section, $r_{\bar{F}}$ is $\bar{\mathcal{F}}$ measurable and the map $\bar{T}$ is a measure-preserving endomorphism of $(\bar{F}, \bar{\mathcal{F}}, \bar{\mu})$.*

Proof. It is a simple exercise to use a thick section to show that $r_{\bar{F}}$ is $\mathcal{F}$-measurable. It is also direct to see that $\bar{T}$ extends to a measurable and measure preserving map on the thick section F. The measurability of this map implies that $\bar{T}^{-1}(\bar{\mathcal{F}}) \subseteq \bar{\mathcal{F}}$. This also shows $\bar{T}$ is measure preserving. □

Theorem 2.3. *Suppose a semiflow T_t has a section $\bar{X}_0$ with $r_{\bar{X}_0} \ge a >$ 0. Then T_t has a section $\bar{X}_1$ with $a \ge r_{\bar{X}_1} \ge a/2$.*

Proof. Set

$$\bar{X}_1 = \{x \in X \mid \text{for some } n \ge 0, T_{an/2}(x) \in \bar{X}_0 \\ \text{but } T_t(x) \notin \bar{X}_0 \text{ for } 0 \le t < an/2\}$$

What $\bar{X}_1$ looks like on an orbit is an arithemetic sequence of points preceding a point in $\bar{X}_0$ spaced at distances $a/2$. This sequence continues back from the point in $\bar{X}_0$ until the next point added would be at or before a previous occurrence of $\bar{X}_0$. A thickening of $\bar{X}_0$ by $a/4$ can be pulled back to form an $a/4$ thickening of $\bar{X}_1$, showing it is a section. □

Definition 2.4. *We say a section $\bar{X}$ of the semiflow T_t is "bounded" if for some values $0 < a < b$, $a < r_{\bar{X}} < b$. Our definition of a section requires $r_{\bar{X}}$ to be bounded below. Hence this is simply asking that the return time also be bounded above.*

Theorem 2.3 has shown that semiflows with sections always possess bounded sections. We will need to know later that we can make the return map large.

Lemma 2.1. *Suppose the semiflow T_t has a section. For any $B > 0$ then it must have a bounded section $\bar{X}$ with $r_{\bar{X}} > B$.*

Proof. It is sufficient here to show that for any endomorphism $\bar{T}$ and $N > 0$, there is a subset $A \subseteq \bar{X}$ that intersects all ergodic components and for which the return time to the subset is $\geq N$. This is just a matter of checking that a proof for $\mathbb{Z}$ actions lifts. Choose a subset B which has positive measure $< 1/N$ for all ergodic components of μ. This is easily done using the Rokhlin decomposition of the space over its invariant algebra. Now set

$$A = \{x \in B \mid r_B(x) \geq N\}.$$

It is easy to see this is a measurable set for the endomorphic action. As $r_A(x) \geq r_B(x)$ we are done.

Thus starting with $\bar{T}$, the return map to some section of thickness α, we can choose a subset as a new section guaranteeing the thickness is at least $N\alpha$ which we make $> B$. We can now fill in to make this new section bounded but still $> B$ using Theorem 2.3. □

We know that all ergodic semiflows possess sections and hence bounded sections. We have not shown this to be true in the nonergodic case. We will not pursue this technical issue here but will assume from here on that the semiflows under consideration are ergodic.

3. Shift Equivalence of Semiflows and Return Maps

Our goal now is to show that virtually all of the classical Kakutani theory of return maps to sections of flows applies to semiflows if one replaces conjugacy with shift equivalence in the appropriate places. We will see that two shift equivalent semiflows will always have shift equivalent return maps. Conversely we will see that if two return maps are shift equivalent and the return times are bounded and agree appropriately then the two semiflows are shift equivalent. This latter fact will allow us to create a generalization for semiflows the "flow-under-a-function" representation of Ambrose and Kakutani. We will not be able to say that any semiflow is conjugate to a semiflow written under a function. We will though be able to say any semiflow is shift equivalent to such a semiflow under a function.

We take a moment to contrast our work with that of Krengel in [11]. Suppose T_t acting on $(X, \mathcal{F}, \mu)$ is a measure preserving semiflow, $\bar{F}$ is a section with return time function $f\colon \bar{F} \to \mathbb{R}^+$. Set $\hat{X} = \{(x,t)\colon x \in \bar{F} \text{ and } 0 \leq t < f(x)\}$. Now the map $(x,t) \to T_t(x)$ factors the flow-under-a-function representation onto the original action T_t. In the automorphic case this map is invertible but in our endomorphic case it is not necessarily. What Krengel does is to pull back to $\hat{X}$ the algebra of measurable sets $\mathcal{F}$ and analyses its structure. He fibers this algebra over the subalgebra of horizontal cuts $\bar{F}_t = \bar{F} \times \{t\} \cap \hat{X}$, giving a family of measure spaces, and characterizes how the flow moves these measure spaces among themselves. We will do something far less subtle. We will show that this factor map from $\hat{X}$ to X is half of a shift-equivalence. We lose something of course as shift equivalence is a much weaker relation than conjugacy. We also will gain a great deal as well as we will be able to lift the theory of induced maps and the natural relation among different sections to the case of semiflows common semiflow. It is worthwhile to ask how one might extend Krengel's structure to the question of how return maps to different cross-sections of a common flow are related.

A little understanding of shift equivalence will be helpful at this point. The definition of shift equivalence is nicely symmetric. We give though simple assymmetirc condition for when the existence of one factor map alone will imply the existence of the other and hence give a shift equivalence.

Lemma 3.1. *Suppose T, S and R are abelian semigroup actions on the spaces X, Y and Z respectively. Suppose ϕ_1 and ϕ_2 factor R onto T and S respectively. Suppose that for some g in the semigroup and a.e. $y \in Y$ we have*

$$T_g(\phi_2(\phi_1^{-1}(\{y\})))$$

is a singleton $\{\phi_3(y)\}$. Then the map $y \to \phi_3(y)$ factors S onto T.

Proof. + The only issue here that is not clear here is that the map ϕ_3 is a measure preserving and measurable map. To see this let m be the joining of the actions T and S given by the factor maps, i.e.

$$m(A \times B) = \mu(\phi_1^{-1}(A) \cap \phi_2^{-1}(B)).$$

Now consider the measure $(\mathrm{id} \times T_g)^* m$. The hypothesis tells us this measure has atomic fibers over the first coordinate space Y. It is standard in the theory of couplings that this implies the map ϕ_3 must be a measure preserving factor map onto the space X. □

Theorem 3.1. *Suppose T and S are abelian semigroup actions on X and Y respectively. Suppose U factors T onto S. Suppose further that for some g in the semigroup and a.e. $y \in Y$ we have $T_g(U^{-1}(\{y\})$ is a singleton $\{V(y)\}$. Then T and S are shift-equivalent by the pair U, V with lag g.*

Proof. Let $\phi_1 = U$ and $\phi_2 = \mathrm{id}$. Now

$$T_g \phi_2 \phi_1(\{y\}) = T_g U^{-1}(\{y\})$$

is a singleton a.s. and hence V factors S onto T. Moreover

$$VU(x) = T_g(x) \text{ and } UV(y) = UT_gU^{-1}(y) = S_g(y),$$

giving the shift-equivalence. □

Theorem 3.2. *Suppose T_t and S_t are shift equivalent semiflows by the factor maps U and V with lag $\ell \geq 0$. Suppose $\bar{X}$ is a section for T_t with return map $\bar{T}$. The set $\bar{Y} = V^{-1}(\bar{X})$ is then a section for S_t the return map for which we write as $\bar{S}$. The maps $\bar{T}$ and $\bar{S}$ are shift equivalent by some maps $\bar{U}$ and $\bar{V}$ where $\bar{V} = V\big|_{\bar{Y}}$ and moreover*

$$r_{\bar{X}}(V(y)) = r_{\bar{Y}}(y).$$

Proof. To begin, let F be an a-thickening of $\bar{X}$, giving a thick section. It is easy to see that $V^{-1}(F) = F'$ must be a thick section, of thickness a, for S_t. It is helpful here to realize that $T_t(V(y)) \in F$ iff $S_t(y) \in F'$ and hence the way $V(y)$ moves in and out of the thick section F gives precisely the same picture as the way y moves in and out of the thick section F'. This tells us immediately that for $\bar{Y}$, the section given by F', $V(\bar{Y}) = \bar{X}$ and

$$r_{\bar{Y}}(y) = r_{\bar{X}}(V(y)) \quad \text{for all} \quad y \in \bar{Y}.$$

We let $\bar{V} = V$ give us one factor map. As $r_{\bar{Y}}(y) \geq a$, letting $n = \left[\frac{\ell}{a}\right]+1$,

$$\bar{S}^n(y) = S_{t(y)}(y)$$

where

$$t(y) = \sum_{j=0}^{n-1} r_{\bar{Y}}(\bar{S}^j(y)) = \sum_{j=0}^{n-1} r_{\bar{X}}(\bar{T}^j(V)y)) := t'(V(y)) > \ell.$$

Thus

$$\bar{S}^n(\bar{V}^{-1}(\{x\})) = S_{t'(x)}S_\ell(\bar{V}^{-1}(\{x\}) = S_{t'(x)}UV(V^{-1}(\{x\})) = \{T_{t'(x)}(x)\}$$

is a singleton. Hence there is a $\bar{U} = \bar{S}^n\bar{V}^{-1}$, completing the shift equivalence. □

We next prove a converse to this, that if two semiflows have shift equivalent sections with equal return time functions then they are shift equivalent. We must require the return times to be bounded for this. It is evident that without this a "natural" argument such as we offer, where the shift equivalence of the semiflows simply extends to the flow lines the matching already made on the sections, could not succeed. No bounded shift forward along flow lines could guarantee to have hit the section and it is only by hitting the section we can force the inverse image to collapse to a singleton.

Theorem 3.3. *Suppose T_t and S_t acting on X and Y respectively are semiflows. Suppose that $\bar{T}$ and $\bar{S}$ are return maps to sections $\bar{X}$ and $\bar{Y}$ of these flows. Further suppose $\bar{T}$ and $\bar{S}$ are shift equivalent with lag n by factor maps $\bar{U}$ and $\bar{V}$. Lastly, suppose that*

$$r_{\bar{Y}}(y) = r_{\bar{X}}(V(y))$$

and that these return maps are bounded. Then T_t and S_t are shift-equivalent by maps U and V where $V(y)$ is on the forward T_t orbit of $\bar{V}(y)$.

Proof. Lifting T_t and S_t to their invertible covers $\hat{T}_t$ and $\hat{S}_t$, the sections $\bar{X}$ and $\bar{Y}$ can be regarded as sections of $\hat{T}_t$ and $\hat{S}_t$ and the return maps $\bar{T}$ and $\bar{S}$ will lift to invertible return maps $\tilde{T}$ and $\tilde{S}$. The map $\bar{U}$ will lift to a conjugacy $\tilde{U}$ between $\tilde{T}$ and $\tilde{S}$ with

$$r_{\tilde{X}}(\hat{x}) = r_{\tilde{Y}}(\tilde{U}(\hat{x})).$$

The classical flow-under-a-function picture of Ambrose and Kakutani then applies to these invertible lifts and we know $\tilde{U}$ extends to a conjugacy $\hat{U}\colon \hat{X} \to \hat{Y}$ between $\hat{T}_t$ and $\hat{S}_t$. Let $\phi_1\colon \hat{X} \to X$ and $\phi_2\colon \hat{Y} \to Y$ be the factor maps of the covering. Thus $\phi_2\tilde{U}\colon \hat{X} \to Y$.

Let B be a bound for $r_{\bar{X}}$ and now for $\hat{x} \in \hat{X}$, set $x = \phi_1(x)$, and for $x \in X$ let $0 \leq \ell(x) \leq B$ be such that $T_{\ell(x)}(x) \in \bar{X}$, i.e. we take the first hit of x on the section $\bar{X}$. Now for $\hat{x} \in \hat{X}$, let $x = \phi_1(\hat{x})$ and

$$\phi_2\hat{U}(\hat{T}_{\ell(x)}(\hat{x})) = \bar{U}\phi_1\hat{T}_{\ell(x)}(\hat{x}) = \bar{U}T_{\ell(x)}(x).$$

We can now calculate that:

$$\begin{aligned}
&S_B(\phi_2\hat{U})(\phi_1)^{-1}(\{x\})\\
&\quad= S_{B-\ell(x)}S_{\ell(x)}\phi_2\hat{U}\phi_1^{-1}(\{x\})\\
&\quad= S_{B-\ell(x)}(\phi_2\hat{U}\hat{T}_{\ell(x)})(\phi_1^{-1}(\{x\}))\\
&\quad= \left\{S_{B_{\ell(x)}}\bar{U}T_{\ell(x)}(x)\right\} \quad \text{is a singleton.}
\end{aligned}$$

Hence we obtain $U = S_B\phi_2\hat{U}\phi_1^{-1}$ as a point map factoring T_t onto S_t.

We now seek a value $L \geq 0$ such that $T_LU^{-1}(\{y\})$ is again a singleton to complete the proof of the existence of the shift equivalence. To begin, suppose that for some value $L' \geq 0$ and some y' we have that $T_{L'}U^{-1}(\{y'\})$ is a singleton $\{x\}$ and moreover $y \in S_{-t}(\{y\}$ and $L > L' + t$. Then

$$\begin{aligned}
T_L(U^{-1}(\{y\})) &= T_{(L-L'-t)}T'_LT_tU^{-1}(\{y\})\\
&\subseteq T_{(L-L'-t)}T_LU^{-1}S_t(\{y\}) \subseteq T_{(L-L'-t)}(\{x\})
\end{aligned}$$

is a singleton.

Next notice that for $\bar{y} \in \bar{Y}$, $\bar{T}^n\bar{U}^{-1}(\{\bar{y}\})$ is a singleton $= \{\bar{V}(\bar{y})\}$. Hence $T_{nB}\bar{U}^{-1}(\{\bar{y}\})$ is a singleton. For such a $\bar{y}$, $U^{-1}(\{\bar{y}\}) = T_{-B}\bar{U}^{-1}(\{\bar{y}\})$ and such $T_{(n+1)B}U^{-1}(\{\bar{y}\})$ is a singleton.

As the return maps are bounded, for any $y \in Y$ there is a value $s(y) < B$ so that $S_{s(y)}(y) := \bar{y} \in \bar{Y}$. Hence for all $y \in Y$, $T_{(n+2)B}U^{-1}(\{y\})$ is a singleton, i.e. using $L = (n+2)B$ will do and we have shown the semiflows are shift equivalent. □

We can now give our version of the flow-under-a-function representation for semiflows. Using Theorem 2.3 we can construct a section $\bar{X}$ with a bounded return time $r_{\bar{X}}$ and return time map $\bar{T}$. Now in $\bar{X} \times \mathbb{R}^+$ we can take the subset $\{(\bar{x}, t) \mid 0 \le t < r_{\bar{X}}(\bar{x})$ and as action put on this set the usual flow under a function using $\bar{T}$ as the return map to the base. Call this constructed flow S_t

Corollary 3.1. *For T_t a measurable and measure preserving semiflow on a standard probability space X, and any section $\bar{X}$ with bounded return times, the associated flow under a function S_t is shift equivalent to T_t.*

Proof. Just note that T_t and S_t share a section, $\bar{T}$ acting on $\bar{X}$, with identical return time map and hence are shift equivalent by Theorem 3.3. □

4. Time Changing Semiflows

It is as reasonable for semiflows as for flows to discuss "time changes". That is say continuous reparametrizations of the time parameter. As usual we view this functionally through a cocycle. Suppose T_t acting on X is a semiflow. Now suppose $q\colon X \times \mathbb{R}^+ \to \mathbb{R}^+$ is measurable and for a.e. x is a continuous bijection of $\mathbb{R}^+$, i.e. a continuous and strictly increasing function. Moreover suppose q is a cocycle in that

$$q(x, t+s) = q(x,t) + q(T_t(x), s).$$

Such a cocycle leads to another flow $S_t(x) = T_{q(x,t)}(x)$ with the same orbits as T_t. S_t inherits a natural invariant measure from μ. Perhaps the easiest way to see this is to take a section for the semiflow T_t, say $\bar{X}$ with return map $\bar{T}$ and return time function $r_{\bar{X}}$. Now make a flow under a function construction over $\bar{T}$ with height function $q(x, r_{\bar{X}}(t))$. Call this semiflow $\hat{S}_t$ and notice that it has $\bar{\mu} \times$ Lebesgue as an invariant measure (perhaps infinite). Now $\hat{S}_t$ factors onto S_t by $(\bar{x}, t) \to S_t(\bar{x})$ and the push-forward of $\bar{\mu} \times$ Lebesgue will be an invariant measure for S_t. It is not difficult to see that up to a normalization this measure is independent of the choice of section.

Definition 4.1. *For a cocycle q as above, we say that q is "integrable" if the measure constructed is finite. We say that it is "bounded" if $q(x,1)$ is bounded above and below on X and "Lipschitz" if for some $C > 1$, a.e. x and all t*

$$t/C \leq q(x,t) \leq Ct$$

Corollary 4.1. *Bounded and Lipschitz cocycles are integrable.*

For flows there is a natural relation between conjugacy of two time changes and cohomology of the cocycles. For semiflows this cannot hold. Again though, if one replaces conjugacy by shift equivalence one gets a natural lift of the result. The map U will be precisely the conjugacy obtained for the flow case. It will be of the form $T_{p(x)}$ and hence unless $p = 0$ cannot be guaranteed invertible.

It is most natural here to change to a different notion of cocycle. For a semiflow S_t with x_2 on the forward orbit of x_1 we set $\alpha_S(x_1, x_2) \in \mathbb{R}^+$ to be that value t where $S_t(x_1) = x_2$. We call such an α an "arrangement" of the orbit. Thus

$$q(x,t) = \alpha_T(x, S_t(x)) \quad \text{and} \quad \alpha_T(x_1, x_2) = q(x_1, S_{\alpha_S(x_1,x_2)}(x_1))$$

relate the cocycle of a time change to the corresponding arrangement.

Definition 4.2. *We say two cocycles q_1 and q_2 are "cohomologous" if the corresponding arrangements α_{S^1} and α_{S^2} satisfy*

$$\alpha_{S^1}(x_1, x_2) = \alpha_{S^2}(x_1, x_2) + p(x_1) - p(x_2)$$

for some $p \colon X \to \mathbb{R}$. We say they are "boundedly cohomologous" if p can be chosen bounded.

Theorem 4.1. *Two cocycles q_1 and q_2 are boundedly cohomologous iff the two constructed time changes S^1 and S^2 are shift equivalent by maps U and V so that $U(x)$ is on the forward orbit of x, i.e. of the form $S^2_{p(x)}(x)$, p bounded.*

Proof. If we lift to the invertible covers for these flows $\hat{S}^i_t$ then it is standard that $U(\hat{x}) = \hat{S}^2_{p(x)}(\hat{x})$ is a conjugacy between $\hat{S}^1$ and $\hat{S}^2$ as

$$\alpha_{S^2}(U(x_1), U(x_2)) = \alpha_{S_1}(x_1, x_2).$$

The function p is only unique up to addition of a constant. Hence we can assume $p \geq 0$ and hence U is well-defined as a factor map from the semiflow S^1 to the semiflow S^2. Replacing p by $p - L$ where L is a bound for p, we obtain $V = S^1_{L-p(x)}(x)$, a factor map from S^2 to S^1 completing the the shift equivalence.

The other direction of the argument is a standard calculation. □

We want to see that the cocycles for a semiflow move in a natural way to other shift-equivalent semiflows.

Theorem 4.2. *Suppose q is a bounded cocycle giving a time-change S_t of the semiflow T_t. Now also suppose T_t is shift-equivalent to some other semiflow T_t^1 by maps U, V. Then $q_1(y,t) = q(V(y),t)$ is a cocycle that gives a time-change of T_t^1 to S_t^1 that is shift-equivalent to S_t by natural factor maps U', V.*

Proof. As U, V give the shift equivalence, we get $q_1(y,t) = q(V(y),t)$. The map V gives us half of the needed shift equivalence to S_t. Now $T_t(V^{-1}(\{x\}) = \{T_{t-L}U(x)\}$ is a singleton for any $t \geq L$, the lag of the shift equivalence. As q is a bounded cocycle we can choose $s > 0$ so that $q(x,s) >$ this lag. Now $S_s(V^{-1}(\{x\}) = \{T_{q(x,s)-L}U(x)\}$ is a singleton, guaranteeing the existence of U' and completing the proof of existence of the shift equivalence. □

So far this work has been quite abstract. The most direct way to see that nontrivial cocycles can be constructed is to use sections. We now use sections to see we can "smooth" bounded cocycles to be Lipschitz in a fashion completely analogous to the invertible case. Our first lemma is very easily argued.

Lemma 4.1. *Suppose That $\bar{T}$ is a measure preserving endomorphism on $\bar{X}$. Suppose further that f_1 and f_2 are two functions bounded both above and away from 0. Let T_t^1 and T_t^2 be the two flows under functions constructed under the graphs of f_1 and f_2 respectively. There is then a Lipschitz time change of T_t^1 that is conjugate to T_t^2.*

Proof. One constructs the map $(x,t) \to \frac{f_2(x)}{f_1(x)}t$ from one flow-under-a-function space to the other. This map conjugates T_t^1 to a semiflow conjugate with the same orbits, and hence within a time change of T_t^2. One easily checks the time change is Lipschitz. It is also not difficult to replace this map by another of the form $(x,t) \to (x, f_x(t))$ giving a C^∞ time change. □

Lemma 4.2. *Suppose S_t is a bounded time change of T_t by the cocycle q. This cocycle is then cohomologous to a Lipschitz cocycle q_1.*

Proof. Let $\bar{X}$ be a bounded section for the flow T_t with return time $r_{\bar{X}} > 1$. Let S_t^1 be the flow under a function constructed over $\bar{T}$ using the function $r_{\bar{X}}$. For any $\bar{x} \in \bar{X}$ let $f(\bar{x}) = q(\bar{x}, r_{\bar{X}}(\bar{x}))$. That the time change is bounded implies that f is bounded both above and below. Let S_t^2 be the flow under a function constructed over the endomorphism $\bar{T}$ with the height function f. We know that S_t^1 and S_t^2 differ by a

Lipschitz time change. On the other hand, S_t^1 is shift equivalent to T_t by Corollary 3.1, and by Theorem 4.2 the Lipschitz time change pulls back to a Lipschitz cocycle q_1 on T. It is not difficult to check that q and q_1 are cohomologous. In fact, for any $\bar{x}$ and $T_t(\bar{x})$ in $\bar{X}$, $q(\bar{x}, t) = q_1(\bar{x}, t)$ and this implies the cohomology. □

We end by lifting the notion of flow equivalence of two flows to the case of semiflows.

Definition 4.3. *We say that two semiflows are "flow shift-equivalent" if a bounded, hence Lipschitz time change of one is shift-equivalent to the other. By Theorem 4.2 this is an equivalence relation.*

We now lift a standard part of the Kakutani theory to the case of semiflows.

Corollary 4.2. *Two semiflows are flow shift-equivalent iff the collection of return maps to bounded sections of one are shift-equivalent to the corresponding return maps of the other and equivalently if one return map to a bounded section of one semiflow is shift equivalent to one such return map of the other.*

Definition 4.4. *We say that two endomorphisms T and S are "flow related" if they arise as return maps to sections of shift-equivalent flows.*

Theorem 4.3. *To be flow related is an equivalence relation. An equivalence class consists of all endomorphisms that are shift-equivalent to some return map to a bounded section of some fixed flow.*

Proof. If $\bar{S}$ arises as a return map to bounded sections of two semiflows S^1 and S^2 then S^1 and S^2 are flow equivalent by Lemma 4.1. Theorem 4.2 now shows us that being flow related is a transitive relation. Symmetry and reflexivity are obvious. Theorem 3.2 allows us to pull shift equivalent versions of an entire class onto one semiflow. □

5. Kakutani Shift Equivalence

The goal of this section is to extend the classical theory of induced maps, and Kakutani equivalence of transformations to the case of endomorphisms. Once more we do this only up to shift equivalence. We expect by now the reader will not find the results surprising as they follow the same format as our earlier work.

For endomorphisms, just as for automorphisms, one can define the induced map on a subset A as the first return map to that subset. This gives a measure preserving map T_A carrying the set to itself. Inducing on a subset for a discrete action is analogous to the construction of a section and return map for a semiflow only technically much simpler. We now list a sequence of results analogous to those of the Section 3 but here for induced maps. The proofs for semiflows work equally well and we will simply indicate which argument from Section 3 applies.

For this work we will assume that T and S are measure-preserving endomorphisms of standard probability spaces X and Y. For a subset $A \subseteq X$ we write $r_A \colon A \to \mathbb{N}$ for the first return time for x to A. We assume that all our actions are ergodic and hence this is a finite value for all $x \in X$. We say T_A is a "bounded" induced map if r_A is bounded. Of course it is enough for r_A to be bounded on A itself.

Theorem 5.1. *Suppose U, V form a shift equivalence between two endomorphisms T and S. For $A \subseteq X$ of positive measure, let $B = V^{-1}(A)$. We conclude that T_A and S_B are shift equivalent by $\bar{V} = V\big|_B$ and some $\bar{U}$. Moreover $r_A(V(y)) = r_B(y)$.*

Proof. The proof is analogous to and simpler than that of Theorem 3.2 □

Theorem 5.2. *Suppose T and S are endomorphisms with subsets A and B such that $\bar{U}, \bar{V}$ form a shift equivalence between T_A and S_B. Moreover suppose for a.e. $y \in B$*

$$r_A(V(y)) = r_B(y)$$

and these return times are bounded. Then T and S are shift equivalent by maps U, V where $V(y)$ is on the forward T orbit of $\bar{V}(y)$.

Proof. This proof is analogous to that of Theorem 3.3. □

Just as one can construct a semiflow under a function from an endomorphism and a return time function to $\mathbb{R}^+$ one can "exduce" an endomorphism T to another endomorphism with return function $f \colon X \to \mathbb{N}$ if $f \in L^1$. As with flows under functions, one takes as measure space $\tilde{X}$ that part of $X \times \mathbb{N}$ the lies under the function,

$$\tilde{X} = \{(x, n) \mid 0 \leq x < f(x)\}.$$

The exduced map T^f is then given by

$$T^f(x, n) = \begin{cases} (x, n+1) & \text{if} \quad n + 1 < f(x), \\ (T(x), 0) & \text{otherwise.} \end{cases}$$

Corollary 5.1. *For T a measurable and measure-preserving endomorphism, if we first induce on a subset A, obtaining T_A and then construct the exduced action using the function $f = r_A$ we obtain an endomorphism $T_A^{r_A}$, and if r_A is bounded then T and $T_A^{r_A}$ are shift equivalent.*

Proof. This proof is analogous to that of Theorem 3.1. □

The classical Kakutani theory [6] tells us that two automorphisms arise as return maps to sections of the same flow iff they induce conjugate actions on subsets and this is equivalent to their exducing conjugate actions via appropriate choices of functions. Our primary goal now is an analog of this for noninvertible actions (Theorem 5.3). As one can choose a variety of ways to replace conjugacy by shift equivalence in these equivalences we get a longer list of equivalent conditions.

Leading up to this, we offer a collection of results. A simple fact often used in the study of automorphisms is that for T an automorphism, T_A and $T_{T^{-1}(A)}$ are conjugate (by T, of course). For endomorphisms, T is only a factor map and so the result fails. The reader will not be surprised to learn though, that one obtains a shift equivalence. This is perhaps the most elementary of all the parallel arguments we have offered.

Lemma 5.1. *For an endomorphism T and a subset of positive measure A, the endomorphisms T_A and $T_{T^{-1}(A)}$ are shift equivalent.*

Proof. The map $U = T$ gives a factoring of $T_{T^{-1}(A)}$ onto T_A. For any $x \in A$,

$$T_{T^{-1}(A)}(U^{-1}(\{x\})) = \{T^{r_{(A)}(x)-1}(x)\}$$

is a singleton. Thus there is a factor V completing the shift-equivalence. □

We add now some very useful lemmas relating shift-equivalence and inducing.

Lemma 5.2. *For a freely acting measure preserving endomorphism T there are disjoint sets $A, B \subseteq X$ with bounded return times.*

Proof. Find a subset C_1 that has measure $< 1/2$ for all ergodic components of the T-invariant measure μ. Hence $r_{C_1} > 1/2$ on a set of positive measure for all ergodic components of μ. Let $C_2 = \{x \in C_1 \mid r_{C_1} > 2\}$, and we will have $r_{C_2} > 2$ uniformly on C_2 and C_2 has positive measure on a.e. ergodic component of μ. Now let

$$A = \{x \in X \backslash C_2 \mid r_{C_2}(x) \text{ is even}\} \text{ and } B = \{x \in X \backslash C_2 \mid r_{C_2}(x) \text{ is odd}\}.$$

Clearly A, B and C_2 form a partition of X. Moreover, $r_A(x) \leq 4$ and $r_B(x) \leq 3$. □

Corollary 5.2. *For a freely acting endomorphism T and subsets A and $B \subseteq X$, both with bounded return times, there are disjoint subsets $A' \subseteq A$ and $B' \subseteq B$ both of which still have bounded return times for T.*

Proof. If $A \cap B$ has measure zero we are done. If not then let $C = A \cap B$ and using the previous lemma find in C two disjoint subsets A'' and B'' that have bounded return times under the action T_C. Now take $A' = (A \setminus C) \cup A''$ and $B' = (B \setminus C) \cup B''$. To see that A' has bounded return times let K be a uniform bound for the return time of A'' under the action of T_C. Among the first K returns of $x \in A'$ to A either some point is in $A \backslash_C$ or all are in C. In this case at least one of these must be in A''. In either case we see a point of A' among these K points in A. The B' argument is identical, of course.

Definition 5.1. *We say two disjoint sets A and B "alternate on orbits" of the endomorphism T if for $x \in A$, $r_B(x) < r_A(x)$ and for $x \in B$, $r_A(x) < r_B(x)$.*

Notice that two sets that alternate on orbits must have equal density on orbits and hence equal measure by the Birkhoff theorem.

□

Lemma 5.3. *Suppose T is a freely acting endomorphism and A and B are subsets with bounded return times. There are then disjoint subsets $A' \subseteq A$ and $B' \subseteq B$ which alternate on T orbits and have bounded return times.*

Proof. The previous corollary tells us we can assume A and B are disjoint. Now set

$$A' = \{x \in A \mid r_B(x) < r_A(x)\} \text{ and } B' = \{x \in B \mid r_A(x) < r_B(x)\}.$$

Visualize the occurrences of A and B on an orbit as coming in blocks of some number > 1 of points of A followed by some number > 1 of points in B etc. The points in A' are the terminal points in each block of points in A (those points which will hit a point of B before they hit another point of A). Similarly B' consists of the terminal points in a block of points in B. It is clear from this that points in these sets alternate on orbits. Suppose now that r_B and r_A are both bounded by K. It is not difficult to see then that $r_{B'}$ and $r_{A'}$ must be bounded by $3K$. □

Lemma 5.4. *If T is an endomorphism and A and B alternate on orbits of T then T_A and T_B are shift equivalent with lag one, i.e. are related by an elementary shift equivalence.*

Proof. Let $U = T^{r_B(x)}(x)$ for $x \in A$ and $V = T^{r_A(x)}(x)$ for $x \in B$. As the sets alternate on orbits we get $VU = T_A$ and $UV = T_B$. The shift equivalence follows from these two as $VUV = VT_B = T_AV$ and $UVU = UT_A = T_BU$. □

Corollary 5.3. *Given T a freely acting endomorphism and subsets A and B with bounded return times, there are $A' \subseteq A$ and $B' \subseteq B$, still with bounded return times and with $T_{A'}$ and $T_{B'}$ shift equivalent.*

We can now pull together our work on endomorphisms and give a clear natural generalization of Kakutani Equivalence for automorphisms to the case of freely acting ergodic endomorphisms.

Theorem 5.3. *The following list of statements about ergodic freely acting measure-preserving endomorphisms are equivalent:*

1) *The endomorphisms T and S induce shift-equivalent actions on subsets with bounded return times.*
2) *The endomorphisms T and S are shift-equivalent to endomorphisms that induce conjugate actions on subsets with bounded return times.*
3) *The endomorphisms T and S arise as induced maps of two shift-equivalent endomorphisms with bounded return times.*
4) *The endomorphisms T and S are shift-equivalent to two endomorphisms that arise as induced maps from some common endomorphism with bounded return times.*
5) *The endomorphisms T and S are flow-related, i.e. are shift equivalent to endomorphisms that arise as return maps to bounded sections of a common flow.*

Proof.

$1 \Leftrightarrow 2$ To see $1 \Rightarrow 2$ suppose T_A and S_B are shift-equivalent. We can pull the return time r_A over to S_B and exduce S_B by this function to produce $\hat{T}$. Now $\hat{T}$ will be shift-equivalent to T by Theorem 5.2. We can also exduce S_B by r_B to obtain $\hat{S}$ that is shift-equivalent to S. Now $\hat{T}$ and $\hat{S}$ both induce S_B. That $2 \Rightarrow 1$ follows directly from Theorem 5.2.

$3 \Leftrightarrow 4$ This once more follows from the fact that the induced maps with bounded return times for shift-equivalent actions are all naturally shift equivalent.

$2 \Leftrightarrow 4$ To see that $2 \Rightarrow 4$, note that 2 tells us T and S are shift-equivalent to maps exduced from a common map R with bounded functions f and g. If we exduce by $\max(f, g)$ we obtain a map that will induce these two exduced maps. To see $4 \Rightarrow 2$ we show $4 \Rightarrow 1$. Suppose R induces maps shift-equivalent to T and S on subsets A and B, with bounded return times. By Lemma 5.3 and Lemma 5.4, T and S induce shift-equivalent maps on subsets with bounded return times. This is statement 1.

$4 \Rightarrow 5$ Suppose R induce maps shift-equivalent to T and S with bounded return times. Build a flow under a function with R as base and height function 1. Now the two induced maps both arise as sections of this flow with bounded return times .

$5 \Rightarrow 1$ Suppose S and T are shift-equivalent to $\bar{S}$ and $\bar{T}$, both of which arise as return maps to sections $\bar{X}$ and $\bar{Y}$ of a flow R_t with both $r_{\bar{X}}$ and $r_{\bar{Y}}$ bounded. We can assume w.l.o.g. that R_t is a flow under a function built over the base map $\bar{T}$.

Between two occurrences of $\bar{X}$ on an R_t orbit there can be at most finitely many occurrences of $\bar{Y}$ and vice versa. For $\bar{x} \in \bar{X}$ set

$$f(\bar{x}) = \#\{t \mid 0 \leq t < r_{\bar{X}}(\bar{x}), \text{ with } R_t(\bar{x}) \in \bar{Y}\}.$$

Exduce the endomorphism $\bar{T}$ by the bounded function f to construct an action $\tilde{T}$ on $\tilde{X}$. We know $\tilde{T}$ induces $\bar{T}$ on a subset with bounded return time. We show that it induces a map conjugate to $\bar{S}$ as well.

List the finitely many elements of the set $\{t \mid 0 \leq t < r_{\bar{X}}(\bar{x}), \text{ with } U_t(\bar{x}) \in \bar{Y}\}$ as

$$0 \leq t_1(\bar{x}) < t_2(\bar{x}) < \cdots < t_{f(\bar{x})}(\bar{x}).$$

Each t_i is a measurable function of $\bar{X}$ where it is defined. As R_t is a flow under a function, the occurrences of points in $\bar{Y}$ are all written precisely as $\{(\bar{x}, t_1(\bar{x})), \ldots (\bar{x}, t_{f(\bar{x})}(\bar{x})\}$. The map $\phi((\bar{x}, t_i(\bar{x}))) = (\bar{x}, i-1)$ maps $\bar{Y}$ 1-1 into the exduced space $\tilde{X}$. The map ϕ conjugates $\bar{S}$ to the map $\tilde{T}$ induces on this subset and the return time to this subset will be bounded.

□

Definition 5.2. *We say two ergodic and freely acting endomorphisms T and S are "Kakutani shift-equivalent" if they satisfy any one of the five equivalent conditions of the previous theorem. We take condition 1 as the formal definition. Having shown condition 5 is an equivalence relation, Kakutani shift equivalence is an equivalence relation.*

Before leaving this subject we add one last result concerning sets that alternate on orbits. Notice that we have seen that in this case the actions are shift equivalent with lag 1. That is to say there is an "elementary strong shift-equivalence" between the two actions. We now show the converse of this fact.

Lemma 5.5. *Suppose $\bar{T}$ and $\bar{S}$ are shift-equivalent endomorphisms with lag 1. They then arise as induced maps from a common endomorphism $\bar{R}$ on sets that alternate on orbits. Equivalently they are* *conjugate to return maps to bounded sections of a common flow which alternate on orbits.*

Proof. Let $\bar{Z}$ be the formal disjoint union of $\bar{X}$ and $\bar{Y}$. Define

$$\bar{R}(z) = \begin{cases} U(z) & \text{if} \quad z \in \bar{X}, \\ V(z) & \text{if} \quad z \in \bar{Y}. \end{cases}$$

□

Corollary 5.4. *Two endomorphisms $\bar{T}$ and $\bar{S}$ are Kakutani shift-equivalent iff they induce maps $\bar{T}_A$ and $\bar{S}_B$ that are conjugate to return maps to bounded sections of some common flow.*

We will discuss the potential of these last two results in the next section.

6. Examples and Questions

We believe it is now quite evident that replacing conjugacy with shift-equivalence as we have done is the natural way to lift the classical theory of sections of flows to semiflows. The shift-equivalences we construct arise in precisely the way the conjugacies do in the invertible theory. The functorial nature of much of the construction makes it evident, we believe, that this is the natural form for the theory.

We give some examples to show that one cannot do better. First we want to give an example of shift equivalent, but not conjugate actions.

Example 1. Let T_1 be the one sided 2-shift, that is to say the left shift on sequences $X = \{0,1\}^{\mathbb{N}}$. Let $T_2 \colon i \to i+1 \mod 6$ on $\{0,1,\dots,5\}$ be a rotation on six points. Take $\hat{X} = X \times \{0,\dots,5\}$ and let $T = T_1 \times T_2$. This is a uniformly 2-to-1 endomorphism. The second coordinate gives its Pinsker algebra.

Set

$$A = \{(x,i) \mid i = 0,3\} \qquad \text{and}$$
$$B = \{(x,i) \mid i = 1,5\}.$$

As A and B alternate on orbits we know T_A and T_B are shift equivalent. On the other hand as T_A is everywhere 8-to-1 but T_B is 16-to-1 on half the space and 4-to-1 on the other half these two actions cannot be conjugate.

On can easily make this example occur on sets in the one sided 2-shift using the Rokhlin lemma.

Example 2 (almost). We now describe a much more rigid example. We will not complete the construction, leaving that to appear elsewhere. Let μ be some shift-invariant and ergodic measure on $X = \{0,1\}^{\mathbb{N}}$ and T be the shift map. In X we represent cylinder sets by strings $v = \{x_i, x_{i+1}, \ldots, x_j\}$ and let v represent all points that have the values x_k for $i \le k \le j$. Define functions $a(n,x)$ for $x \in X$ and $n \in \mathbb{N}$ by

$$a(n,x) = E_\mu(\{x_0, x_1, \ldots, x_{n-1}\} \mid T^{-n}(\mathcal{F}))(x).$$

That is to say, $a(n,x)$ is the conditional expectation that one sees the values $x_0, \ldots, x_{n-1}$ at the first n positions of a point conditioned on all the remaining values. Notice that for the standard two shift-these values are always 2^{-n} and if a measure gives T entropy zero, then $a(n,x) = 1$ a.s. We now define an extremely different circumstance.

Definition 6.1. *We say a T invariant measure μ on $X = \{0,1\}^{\mathbb{N}}$ is "Kakutani rigid" if on a subset of X of full measure $a\colon \mathbb{N} \times X \to [0,1]$ is 1-to-1. That is to say, the values $a(n,x)$ determine both n and x a.s.*

We will not show here that Kakutani rigid measures exist. We will show more elsewhere, that in fact for any $h > 0$ among measure μ with $h_\mu(T) \ge h$ the Kakutani rigid measures form a residual set weak*.

Theorem 6.1. *Suppose μ is an ergodic and Kakutani rigid measure on $\{0,1\}^{\mathbb{N}}$. Then w.r.t. μ, two induced maps T_A and T_B are conjugate iff $A = B$, and the only available conjugacy is the identity.*

Proof. If ϕ is the conjugacy of T_A to T_B then we must have

$$a(r_A(x), x) = a(r_B(\phi(x)), \phi(x))$$

which can only hold if $x = \phi(x)$. □

Questions.

A Williams conjecture for endomorphisms. The results at the end of Section 5 lead to a very natural question. We have used shift equivalence as the natural equivalence relation with which to replace conjugacy. One could make another choice. We saw in those results that an elementary shift equivalence leads to stronger conclusions. There are two ways to turn the relation of differing by an elementary equivalence into an equivalence relation. One is shift equivalence. The other is strong shift equivalence, the minimal equivalence relation containing the elementary shift equivalences. That is to say, two semigroup actions are "strong shift equivalent" if they differ by a finite sequence of elementary shift equivalences. In the context of subshifts of finite type this led to the Williams conjecture [18] [16] and the deep work culminating in the counterexample of Kim and Roush [?]. We are led to ask in the context of measure preserving semiflows and endomorphisms whether shift equivalence implies strong shift equivalence.

Even Kakutani shift-equivalence. It has become natural to refine Kakutani equivalence by a somewhat stronger equivalence relation, called “even Kakutani equivalence”, where the subsets on which one induces have equal measures. This equivalence relation (it is one) can be described as a restricted orbit equivalence as defined in [7]. This leads one to ask to what degree restricted orbit equivalence can be lifted to a notion of orbit shift-equivalence for endomorphisms. We note that for endomorphisms the forward orbits do not form equivalence classes. One normally considers the equivalence relation of lying on the “full orbit” when considering endomorphisms endomorphisms. Two points x_1 and x_2 are on the same full orbit if for some i and j, $T^i(x_1) = T^j(x_2)$. For restricted equivalence relations like even Kakutani equivalence, where the order structure of the orbit is essentially preserved, one can perhaps work with the directed relation of lying on the same forward orbit. As a suggestion, one can define a notion of “orbit shift-equivalence” to be the existence of two measure preserving maps U and V such that U(a forward orbitof T) is all but finitely many elements of a forward orbit of S, and symmetrically V(a forward orbit of S) is all but finitely many elements of a forward orbit of T and VU maps a.e. forward orbit of T into itself. As an alternative notion, one could ask that U and V images of a forward orbit cover subsets of density one of the image forward orbits. Katok [8] and Sataev [15] and Ornstein and Weiss [13] showed that one could lift the ideas of the Ornstein isomorphism theory to the study of even Kakutani equivalence. This has been given broad generalization in [7]. It is known that the Ornstein isomorphism theory of Bernoull shifts can be lifted to uniformly p-to-1 endomorphisms [4]. The non-uniform case is understood as well, having been studied by Peter Jong in his thesis [5]. To what degree can one find a parallel “loosely Bernoulli” theory for endomorphisms?

A two-valued representation. Arques and Gabriel [3] have extended the work of Krengel [11] and Rudolph [14] by showing that a cross-section can be chosen for any ergodic semiflow so that the return time function takes on only two values. One normally takes 1 and α for the two values, where α is irrational, but they are arbitrary subject to being irrationally related. Even Kakutani equivalence can be strengthened in this case to allow one to characterize those transformations that arise as sections of a common flow where the return times take on only the values 1 and α. This strengthened relation for automorphisms is again a restricted orbit equivalence (see [7]). If one is successful in “loosely Bernoulli” theory for endomorphisms as we sketched above, does this relation also give rise to such a theory? Arques and Gabriel [2] have also shown that, subject to the natural entropy constraint, the two sets under the values 1 and α can be made to generate the two-sided extension. It would be impossible for this two-set partition to be chosen to generate one-sidedly. Perhaps though one might be able to select this partition so that it generates a system that is shift-equivalent to the original. Once more this would have to be subject to the semiflow not having more entropy than such a partition can support.

Uniformly dyadic actions. The counterexamples we have given work by controlling the conditional expectations of the future conditioned on the past. Is this the only way such examples can occur? Can one construct uniformly dyadic actions (actions that are everywhere 2-to-1 and for which the conditional expectations of the two inverse images are always $(1/2, 1/2)$) that are shift-equivalent but not conjugate? Even more difficult, can such be made that are Kakutani shift-equivalent but not conjugate?

Larger semigroups. Katok [9] has shown how to lift the theory of sections and Kakutani equivalence to actions of $\mathbb{R}^n$ in particular. Commuting endomorphisms arise naturally as, for example, multiplication by p and q, mod 1. Can the theory of sections be lifted in a similar fashion to higher dimensional actions? It is not enough here to just find points on an $\mathbb{R}^n$ orbit that are separated. The Katok theory shows that for n-dimensionsal flows, the points of the section can be chosen to lie on an approximate $\mathbb{Z}^n$ lattice and carry a natural $\mathbb{Z}^n$ action. Obtaining this refined a picture seems difficult.

Reverse filtrations. The study of one sided actions is particularly natural in probability theory. Conjugacies and Kakutani equivalences for automorphisms require one have information about the arbitrarily distant future, a unrealistic situation. Shift-equivalence offers an interesting middle ground here. Suppose we have two endomorphisms given as shift maps on one sided stationary sequences of random variables X_i and $Y_i, i \in \mathbb{N}$. What the existence of a shift-equivalence tells us is that X_i determines Y_i but we cannot code X_i back again from Y_i until we see some finite amount of the future of Y_i. One natural direction this leads to as the following equivalence relation between reverse filtrations (decreasing families of σ-fields). We say two such $\mathcal{F}_i$ and $\mathcal{G}_i$

are "shift-equivalent" if there is a measure preserving map $\phi\colon X \to Y$ so that for some $\ell \geq 0$,

$$\mathcal{F}_i \supseteq \phi^{-1}(\mathcal{G}_i) \supseteq \mathcal{F}_{i+\ell}.$$

Calling ℓ the "lag" of the shift-equivalence, one can again define an elementary equivalence to be one with lag 1, strong shift equivalence as the minimal equivalence relation it generates and formulate a Williams conjecture asking whether shift equivalence implies strong shift equivalence.

References

1. Kakutani S. Ambrose, W., *Structure and continuity of measurable flows*, Duke Math J. **9** (1942), 25–42.
2. Didier Arques and Patrick Gabriel, *Sur la représentation de Rudolph des flots filtrés ergodiques*, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 10, A551–A554.
3. ______, *Théorème de Rudolph filtré; partition génératrice*, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 21, A1393–A1396.
4. C. Hoffman and D.J. Rudolph, *Uniform endomorphisms which are isomorphic to a bernoulli shift*, Annals of Math. **156** (2002), 79–101.
5. P. Jong, *On the isomorphism problem of p-endomorphisms*, Ph.D. thesis, Univ. of Toronto, 2003.
6. S. Kakutani, *Induced measure preserving transformations*, Proc. Imp. Acad. Tokyo **19** (1943), 635–641.
7. J. W. Kammeyer and D. J. Rudolph, *Restricted orbit equivalence for actions of discrete amenable groups*, Cambridge University Press, 2001.
8. A. Katok, *Monotone equivalence in ergodic theory*, Math. USSR Isvestijor **11, no. 1** (1977), 99–146.
9. ______, *The special representation theorem for multi-dimensional group actions*, , Dynamical Systems, I. Warsaw, Asterisque **49** (1977), 117–140.
10. K.H. Kim and F.W. Roush, *Williams conjecture is false for irreducible subshifts*, Annals of Math. **149** (1999), no. 2, 545–558.
11. U. Krengel, *Darstellungssätze für Strömungen und Halbströmungen. I*, Math. Ann. **182** (1969), 1–39.
12. I. Kubo, *Quasi-flows*, Nagoya Math. J. **35** (1969), 1–30.
13. Rudolph D.J. andf Weiss B. Ornstein, D.S., *Equivalence of measure preserving transformations*, Memoirs of the AMS **37** (1982), no. 262.
14. D.J. Rudolph, *A two-valued step coding for ergodic flows*, Math. Zeit. **150** (1976), 201–220.
15. E. Sataev, *An invariant of monotone equivalence determining the quotients of automorphisms monotonely equivalent to a bernoulli shift*, Math. USSR Izvestija **11, no. 1** (1977), 147–169.
16. J.B. Wagoner, *Strong shift equivalence theory and the shift equivalence problem*, Bull. Amer. Math. Soc. **36** (1999), no. 3, 271–296.
17. R.F. Williams, *Classification of one dimensional attractors*, Global Analysis, Proc. Sympos. Pure Math., 1968 (Berkeley Calif.), vol. XIV, AMS, 1970, pp. 341–361.

18. ______, *Classification of subshifts of finite type*, Ann. of Math. **98** (1973), no. 2, 120–153, errata, ibid. (2), **99**, (1974), 380–381.