1. [15 pts] Let $L^{ul} = \{ R \}$ where $R$ is a binary relation symbol, and let $A_1 = (\omega, \leq)$, $A_2 = (\mathbb{Z}, \leq)$, $A_3 = (\mathbb{Q}, \leq)$ and $A_4 = (\omega \setminus \{0\}, |)$. For each $1 \leq i < j \leq 4$ find a sentence $\sigma_{i,j}$ of $L$ such that $A_i \models \sigma_{i,j}$ but $A_j \models \neg \sigma_{i,j}$.

2. [20 pts] Working directly from the definitions, either prove or find a counterexample to each of the following:
   
   a) $\models [(\exists x P x \to \exists x Q x) \to \exists x (P x \to Q x)]$
   
   b) $\models [\exists x (P x \to Q x) \to (\forall x P x \to \exists x Q x)]$
   
   c) $\models (\forall x P x \to Q y) \to (P y \to \exists x Q x)$

3. [15 pts] Fix a sequence $\{\theta_i\}_{i \in \omega}$ of sentences of $L$. For any sentence $\sigma$ of sentential logic let $\sigma^*$ be the result of replacing all occurrences of $S_i$ in $\sigma$ by $\theta_i$ for each $i \in \omega$.
   
   a) Give a definition by recursion of $\sigma^*$.
   
   b) Define, for each $L$-structure $A$, a truth assignment $h_A$ such that for every $\sigma$ of sentential logic we have $h_A \models \sigma$ if $A \models \sigma^*$. (Note that it follows that $\sigma^*$ is valid whenever $\sigma$ is a tautology)

4. [15 pts] Show the following hold, using the rules established in the notes.
   
   a) $\vdash (\forall x P x \to \exists x Q x) \to \exists x (P x \to Q x)$
   
   b) $\vdash \exists x (P x \to \forall y P y)$

5. [15 pts] Given $\Sigma_1, \Sigma_2 \subseteq S n_L$ define $\Sigma^* = \{ \theta \in S n_L : \Sigma_1 \models \theta$ and $\Sigma_2 \models \theta \}$. Prove that for every $L$-structure $A$, $A \models \Sigma^*$ iff either $A \models \Sigma_1$ or $A \models \Sigma_2$.

**NOTE:** Your solutions must include enough detail to justify your conclusions.