1. [15 pts] (a) Let $T$ be a complete theory of $L$, and let $\Phi(x)$ and $\Psi(y)$ be types each of which is realized on some model of $T$. Prove that $T$ has some model realizing both $\Phi$ and $\Psi$.

[5 pts] (b) Give an example to show that the result in part (a) can fail if the theory $T$ is not complete.

2. [20 pts] Let $\mathfrak{A} = (\mathbb{Q}, +, \cdot, 0, 1)$. Prove that $T = Th(\mathfrak{A})$ is not $\omega$-categorical.

3. [20 pts] Let $\Phi(x)$ be a type consistent with the complete theory $T$ but which is realized by at most one element in every model of $T$. Prove that there is some formula $\psi(x)$ consistent with $T$ such that $T \models \forall x(\psi(x) \rightarrow \varphi(x))$ for all $\varphi \in \Phi$.

4. [20 pts] Let $T$ be a complete theory of $L$, and let $\mathfrak{A}$ and $\mathfrak{B}$ be models of $T$. Prove that there is some $\mathfrak{C} \models T$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ can be elementarily embedded in $\mathfrak{C}$.

NOTE: Your solutions must include enough detail to justify your conclusions.