CENTRAL LIMIT THEOREM FOR EXCITED RANDOM WALK IN THE RECURRENT REGIME

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Let $\mathcal{Y} = \mathbb{Z}\times(\mathbb{Z}/L\mathbb{Z})$, where $L > 1$ is an integer, $G = \{-e_1, e_1, -e_2, e_2\}$ where e_j are coordinate vectors. We denote the coordinates of points $y \in \mathcal{Y}$ by $(x(y), s(y))$. Consider a cookie environment on $\mathcal{Y}$, that is, for each $y \in \mathcal{Y}$, $j \in \mathbb{N}$, there is a probability distribution $\omega(y, j, e)$ on G. Consider an excited random walk $Y_n = (X_n, S_n)$ that is

$$\mathbb{P}(Y_{n+1} - Y_n = e|Y_1 \dots Y_n) = \omega(Y_n, l_n, e)$$

where l_n is the number of visits to Y_n by time n. (We denote by $\mathbb{P}$ and $\mathbb{E}$ the quenched probability and expectation with fixed ω and by $\mathbf{P}$ and $\mathbf{E}$ the annealed probability and expectation.) Y_n is called (multi-)*excited random walk* (ERW). We make the following assumptions:

(A) $\delta(y, j) := \omega(y, j, e_1) - \omega(y, j, -e_1) \geq 0$,

(B) $\exists \kappa > 0$ such that $\omega(y, j, e) \geq \kappa$,

(C) ω is stationary with respect to G-shifts and ergodic.

(D) Let $\delta(y) = \sum_{j=1}^{\infty} \delta(y, j)$ then

$$\delta := \mathbf{E}(\delta(y)) < \frac{1}{L}.$$

(E) For each $\varepsilon > 0$ there exists $N(\varepsilon, y)$ such that for each $j \geq N$, for each $e \in G$ $|\omega(y, j, e) - \frac{1}{4}| < \varepsilon$. Moreover $\mathbf{E}(N(\varepsilon, y)) < \infty$.

The quantity δ introduced in (D) plays a crucial role in description of the behavior of ERW. In particular Y_n is recurrent iff $\delta L \leq 1$ ([7, 8]). Several papers addressed the limiting behavior of the ERW in the transient regime [5, 1, 2, 4]. Our paper deals with recurrent ERW.

Let $\mathcal{B}(t)$ denote the Brownian motion with variance $\frac{t}{2}$. Recall ([3]) that for all $\alpha, \beta < 1$ and for almost every realization of $\mathcal{B}$ there exists unique solution $\mathcal{W}(t)$ of the equation

$$\mathcal{W}(t) = \mathcal{B}(t) + \alpha \max_{[0,t]} \mathcal{W}(s) + \beta \min_{[0,t]} \mathcal{W}(s) \tag{1}$$

which is called (α, β)*-perturbed Brownian Motion.*

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Theorem 1. *As* $n \to \infty$ $\mathcal{W}_n(t) := \frac{X_{[nt]}}{\sqrt{n}}$ *converges to* (α, β)*-perturbed Brownian Motion where* $\alpha = -\beta = \delta L$.

Remark. *A similar result is valid for ERW on* $\mathbb{Z}$ *with obvious modifications. Namely,* $G = \{-e, +e\}$, *condition (E) becomes* $|\omega(y, j, e) - \frac{1}{2}| < \varepsilon$ *and the variance of the limiting Brownian Motion equals* t.

Remark. *Our result leaves open the critical case* $\delta L = 1$. *(Observe that* (1) *is not well posed if* $a = 1$.*)*

We divide the proof into several steps. Fix $T > 0$.

Lemma 1. *For any* $\varepsilon > 0$ *there exists* $R > 0$ *such that for large* n

$$\mathbb{P}(\min_{[0,T]} \mathcal{W}_n(t) < -R) \leq \varepsilon.$$

Proof. Denote

$$\Delta_k = X_{k+1} - X_k, \quad \bar{\Delta}_k = \mathbb{E}(\Delta_k | Y_1 \dots Y_k) = \delta(Y_k, l_k),$$

$$C_n = \sum_{k=0}^{n-1} \bar{\Delta}_k, \quad B_n = \sum_{k=0}^{n-1} \left[\Delta_k - \bar{\Delta}_k\right].$$

By assumption (A), $X_n \geq B_n$. For a process Z_n let $\tau(z, Z)$ be the first time $Z_\tau = z$. We need to show that $P(\tau(-R\sqrt{n}, X) \leq Tn) \leq \varepsilon$ provided that R and n are large enough. We have $\tau(-R\sqrt{n}, X) \leq \tau(-R\sqrt{n}, B)$. Since B_n is a martingale whose quadratic variation grows at most linearly, it follows (see e.g. [6], Section IV.4) that for each $m \in \mathbb{N}$ there is a constant γ_m such that

$$\mathbb{P}(\max_{k \leq n} B_k \geq \xi\sqrt{n}) \leq \frac{\gamma_m}{(1+\xi)^m}. \tag{2}$$

which implies the result we need. □

Lemma 2. *For any* $\varepsilon > 0$ *there exists* $R > 0$ *such that for large* n

$$\mathbb{P}(\max_{[0,T]} \mathcal{W}_n(t) > R) \leq \varepsilon.$$

Proof. Let R_1 be such that

$$\mathbb{P}(\min_{[0,T]} \mathcal{W}_n(t) < -R_1) \leq \frac{\varepsilon}{2}.$$

We shall show that if $R \gg R_1$ then for large n

$$\mathbb{P}(\min_{[0,T]} \mathcal{W}_n(t) \geq -R_1, \max_{[0,T]} \mathcal{W}_n(t) > R) < \frac{\varepsilon}{2}.$$

Indeed by the ergodic theorem if $-R_1\sqrt{n} \leq X_k \leq R\sqrt{n}$ for $k \leq m$ then for each $\eta > 0$ we have

$$C_m < (R + R_1 + \eta)\delta L\sqrt{n}.$$

provided that n is large enough. Since $X = B + C$ this implies that

$$\mathbb{P}(\tau(R\sqrt{n}, X) \leq Tn, \tau(-R_1\sqrt{n}, X) \geq Tn) \leq \mathbb{P}(\tau([R-(R+R_1+\eta)\delta L]\sqrt{n}, B) \leq Tn).$$

Since $\delta L < 1$, the value of $R - (R + R_1 + \eta)\delta L$ can be made as large as we wish by choosing R large and so the last expression can be made as small as we wish due to (2). ☐

Let $r_n = \max_{k\leq n}(X_k) - \min_{k\leq n}(X_k)$ denote the range of the walk.

Lemma 3. *For almost every ω the following holds. As $n \to \infty$ the process $\mathcal{B}_n(t) := \frac{B_{[nt]}}{\sqrt{n}}$ converges weakly to $\mathcal{B}(t)$.*

Proof. Since B_n is martingale it suffices to show that

$$\sup_{t\in[0,T]} \left| \frac{V_{[nt]}}{n} - \frac{t}{2} \right| \to 0$$

where V_n is the quadratic variation of B_n. For the discrete time process it is enough to show that for almost every ω

$$\max_{0\leq m\leq n} \left| \frac{V_m}{n} - \frac{m}{2n} \right| \to 0 \text{ in probability.}$$

Fix $\varepsilon > 0$. Choose N_0 such that

$$\mathbf{E}([N(\varepsilon, y) - N_0]^+) < \varepsilon \tag{3}$$

where $N(\varepsilon, y)$ is a constant from condition (E). Split $V_m = V_m^- + V_m^+$ where

$$V_m^- = \sum_{k=0}^{m-1} \left[\Delta_k - \bar{\Delta}_k\right]^2 I(l_k \leq N_0), \quad V_m^+ = \sum_{k=0}^{m-1} \left[\Delta_k - \bar{\Delta}_k\right]^2 I(l_k > N_0).$$

Then $V_m^- \leq 4N_0 L r_m \ll n$ (by Lemmas 1 and 2) whereas

$$V_m^+ = \frac{m}{2} + \epsilon'_m + \epsilon''_m$$

where ϵ'_m comes from the terms with $l_k > N(\varepsilon, Y_k)$ and so $|\epsilon'_m| \leq 2\varepsilon n$ and ϵ''_m comes from the terms with $N_0 < l_k \leq N(\varepsilon, Y_k)$ and so by (3) and ergodicity $|\epsilon''_m| \leq \varepsilon L r_n$ for large n. ☐

Lemma 4. *$\{\mathcal{W}_n(t)\}$ is tight.*

Proof. Since $\mathcal{W}_n(0) = 0$ by Arcela-Ascoli Theorem it is sufficient to show that for each $\varepsilon, \eta > 0$ there are n_0 and $\kappa > 0$ such that for $n \geq n_0$.

$$\mathbb{P}(\sup_{0\leq t_1\leq t_2\leq T, t_2-t_1<\kappa} |\mathcal{W}_n(t_2) - \mathcal{W}_n(t_1)| \geq \eta) \leq \varepsilon \tag{4}$$

Choose R so that $\mathbb{P}(\max_{[0,T]} |\mathcal{W}_n(t)| > R) \leq \frac{\varepsilon}{2}$. Consider a finite set $\mathcal{H} = \{\frac{m\eta}{2}\}_{|m|\leq 2R/\eta}$. To establish (4) it suffices to show that $\mathbb{P}(A_\eta) \leq \frac{\varepsilon}{2}$ where

$$A_\eta = \{\exists 0 \leq t_1 \leq t_2 \leq T : t_2 - t_1 < \kappa \text{ and } \mathcal{W}_n(t_1) \neq \mathcal{W}_n(t_2), \quad \mathcal{W}_n(t_j) \in \mathcal{H}\}.$$

By Lemma 3 $\{\mathcal{B}_n(t)\}$ is tight. Consequently for any η^* there exists κ such that $\mathbb{P}(D_\eta) \leq \frac{\varepsilon}{4}$ where

$$D_\eta = \{\max_{0\leq t_1,t_2\leq T, |t_1-t_2|\leq\kappa} |\mathcal{B}_n(t_1) - \mathcal{B}_n(t_2)| \geq \eta^*\eta\}.$$

Now the argument of Lemmas 1 and 2 shows that if $\eta^* \ll 1$ then

$$\mathbb{P}(A_\eta D_\eta^c) < \frac{\varepsilon}{4}.$$

Namely we first show that on D_η^c if $\mathcal{W}_n(t_1) \in \mathcal{H}$ then

$$\mathbb{P}(D_\eta^c \text{ and } \min_{[t_1,t_1+\kappa]} \mathcal{W}_n(s) < \mathcal{W}_n(t_1) - 2\eta\eta^*)$$

is small and then that

$$\mathbb{P}\left(D_\eta^c \text{ and } \min_{[t_1,t_1+\kappa]} \mathcal{W}_n(s) \geq \mathcal{W}_n(t_1) - 2\eta\eta^* \text{ and } \max_{[t_1,t_1+\kappa]} \mathcal{W}_n(s) \geq \mathcal{W}_n(t_1) + \frac{\eta}{2}\right)$$

is small. □

Let T denote the space shift $(T^k\omega)((x,s),j,e) = \omega((x+k,s),j,e)$ and

$$Z(a,b) = \sum_{(x,s):a\leq x\leq b} \delta(x,s)$$

denote the total amount of cookies stored between a and b. Throughout the rest of the paper we shall write τ_x instead of $\tau(x,X)$ since the second argument is going to be fixed.

The next lemma is a slight variation on [8].

Lemma 5. *For each N, ε there exists a number M and a set Ω_M such that $\mathbf{P}(\Omega_M) > 1-\varepsilon$ and if $T^x\omega \in \Omega_M$ then for each s*

$$\mathbb{P}(Y_n \textit{ visits } (x,s) \textit{ at least } N \textit{ times before } \tau_{x+M}) \geq 1-\varepsilon. \tag{5}$$

Proof. By ellipticity (condition (B)) it is enough to prove the result with (5) replaced by

$$\mathbb{P}(X_n \text{ visits } x \text{ at least } N \text{ times before } \tau_{x+M}) \geq 1-\varepsilon.$$

Let $\tilde{\tau}_m$ be the first time strictly greater than τ_x when either $|X_{\tilde{\tau}} - x| = m$ or $X_{\tilde{\tau}} = x$. Pick two numbers p, p' such that $\delta L < p' < p < 1$. We claim that if m_1 is large enough then for most environments

$$\mathbb{P}(X_{\tilde{\tau}_{m_1}} = x) > 1-p. \tag{6}$$

There are two cases to consider: $X_{\tau_x+1} = x+1$ and $X_{\tau_x+1} = x-1$ (the case $X_{\tau_x+1} = x$ is trivial). We consider the first case (the second case is easier).

$$\mathbb{P}(X_{\tilde\tau_{m_1}} = m_1 | X_{\tau_x+1} = x+1) = \frac{\mathbb{E}(C_{\tilde\tau_{m_1}} - C_{\tau_x}) + 1}{m_1} \leq \frac{Z(x, x+m_1) + 1}{m_1}.$$

So (6) holds if $Z(x, x+m_1) < m_1 p'$ (observe that we need not impose any restrictions in case $X_{\tau_x+1} = x-1$). Next

$$\mathbb{P}(X_{\tilde\tau_{m_2}} = m_2 | X_{\tilde\tau_{m_1}} = x+m_1) = \frac{\mathbb{E}(C_{\tilde\tau_{m_2}} - C_{\tilde\tau_{m_1}}) + m_1}{m_2} \leq \frac{Z(x, x+m_2) + m_1}{m_2}.$$

Thus if $m_2 \gg m_1$ and $Z(x, x+m_2) < p' m_2$ then

$$\mathbb{P}(X_{\tilde\tau_{m_2}} = x + m_2 | X_{\tilde\tau_{m_1}} = x + m_1) < p.$$

Thus if both $Z(x, x+m_1) < p' m_1$ and $Z(x, x+m_2) < p' m_2$) then

$$\mathbb{P}(X_{\tilde\tau_{m_2}} = x + m_2) < p^2.$$

Continuing we construct a sequence $\{m_k\}$ such that on $\bigcap \{Z(x, x + m_j) < p' m_j\}$ we have

$$\mathbb{P}(X_{\tilde\tau_{m_k}} = x + m_k) < p^k.$$

Thus on this set

$$\mathbb{P}(X \text{ returns to } x \text{ before } \tau_{x+m_k}) \geq (1 - p^k).$$

Since the amount of cookies between x and $x + m_j$ only decreases between the returns the same argument shows that

$$\mathbb{P}(X \text{ returns to } x \text{ at least } N \text{ times before } \tau_{x+m_k}) \geq (1 - p^k)^N.$$

If k is large enough this number can be made as close to 1 as we need. This proves the lemma with $M = m_k$ and $\Omega = \bigcap_{j=1}^k \{Z(0, m_j) \leq p' m_j\}$. □

Lemma 6. *For almost all* ω, $\frac{C_n - \alpha r_n}{r_n} \to 0$ *in probability.*

Proof. Let $\varepsilon > 0$. Take N such that

$$\sum_{j=N+1}^{\infty} \mathbf{E}(\delta(y, j)) < \varepsilon.$$

Split $C_n = C_n^- + C_n^+$, where

$$C_n^- = \sum_k \bar\Delta_k I(l_k \leq N), \quad C_n^+ = \sum_k \bar\Delta_k I(l_k > N).$$

By ergodicity for large n we have $|C_n^+| \leq 2\varepsilon r_n$ so the main contribution comes from C_n^-. Next

$$C_n^- = \sum^{*} \sum_{j=1}^{N} \delta(y,j) I(Q(y,j,n))$$

where the summation in (*) runs over y with

$$\min_{k\leq n}(X_k) \leq x(y) \leq \max_{k\leq n}(X_k)$$

and $Q(y,j,n)$ is the event that Y visits y at least j times before time n. Take the large M and split $C_n^- = C_n^\partial + C_n^i$ where C_n^∂ contains the terms $y=(x,s)$ where x is within distance M from either maximum or minimum of $X_k, k \leq n$ and C_n^i contains the remaining terms. Then $C_n^\partial = O(M)$ (namely by condition (D) $C_n^\partial \leq 2M$). On the other hand

$$C_n^i = \sum_y \sum_{j=1}^{N} \delta(y,j) - \sum_y \sum_{j=1}^{N} \delta(y,j) I(Q^c(y,j,n)).$$

Due to ergodicity the first term here is asymptotic to

$$[L \sum_{j=1}^{N} \mathbf{E}(\delta(y,j))] r_n$$

and the term in brackets is within ε from α. The second term is less than

$$\hat{C}_n = \sum_y \sum_{j=1}^{N} I(\hat{Q}(y,j,M))$$

where $\hat{Q}((x,s),j,M)$ is the event that the j-th visit to (x,s) occurs after time τ_{x+M}. We claim that $\hat{C}_n = o(r_n)$ in L^1 and hence in probability. Indeed

$$\mathbb{E}(\hat{C}_n) = \sum_y \sum_{j=1}^{N} \mathbb{P}(\hat{Q}(y,j,M)) \leq L \sum_{x=1}^{r_n} (I(T^x\omega \notin \Omega_M) + \varepsilon)$$

due to Lemma 5. By ergodicity this expression can be made smaller than $2L\varepsilon r_n$ by choosing M large. □

Proof of theorem 1. Let $\mathcal{W}(t)$ denote a weak limit of $\mathcal{W}_n(t)$. By Lemmas 3 and 6 $\mathcal{W}(t)$ satisfies (1) and we are done by [3]. □

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