LIVSIĈ THEORY FOR COMPACT GROUP EXTENSIONS OF HYPERBOLIC SYSTEMS.

DMITRY DOLGOPYAT

Dedicated to Yu. S. Ilyashenko on the occasion of his 60th birthday.

Abstract. We prove Livsiĉ type results for rapidly mixing compact group extensions of Anosov diffeomorphisms.

1. Introduction.

Recently several new phenomena in dynamics were discovered by looking at small perturbations of compact group extensions of hyperbolic systems [8, 14]. In view of this it is desirable to develop a general theory of perturbations of such systems. The first step towards this goal is to understand infinitesimal perturbations that is to study homological equations over such systems. In this note we study regularity of solutions to cocycle equations. Regularity theory plays an important role in rigidity theory. Two of the most studied cases are translations of $\mathbb{T}^d$ and Anosov diffeomorphisms (see [7, 11, 15] for the analysis of some other systems). The systems considered in our paper exhibit a mixture of hyperbolic and elliptic behaviors.

Let M be a compact C^∞ Riemannian manifold and $f : M \to M$ be an Anosov diffeomorphism. Let G be a compact connected Lie group, H a Lie subgroup of G, $Y = G/H$ and $\tau \in C^\infty(M, G)$. Let $N = M \times Y$. Define $F : N \to N$ by $F(x, y) = (fx, \tau(x)y)$. We say that a function A on N is a coboundary if

$$A = B - B \circ F. \tag{1}$$

If B is bounded, Hölder, smooth etc. we say that A is a bounded, Hölder, smooth etc. coboundary. Let $\phi \in C^\alpha(M)$, μ_ϕ be the Gibbs state with potential ϕ and $d\nu_\phi = d\mu_\phi dy$.

In this note we prove the following. Fix $z_0 \in N$.

Theorem 1. *Let F be rapidly mixing. Let $A \in C^\infty(N)$ be a coboundary in $L^2(\nu_\phi)$ for some Hölderfunction ϕ. Then A is C^∞ coboundary. In*

2000 *Mathematics Subject Classification.* 3730C, 37D30, 37J40.

Key words and phrases. cocycle equation, transfer operator, partial hyperbolicity, small divisors.

particular, if A is a bounded coboundary then it is a C^∞ coboundary. Moreover there exists k_0 such that if A belongs to $C^k(N)$ then B belongs to $C^{k-k_0}(N)$. If B satisfies a normalization condition

$$B(z_0) = 0$$

then

$$||B||_{k-k_0} \leq \text{Const}||A||_k. \tag{2}$$

We refer the reader to the next section for the definition of rapid mixing. Recall ([4]) that it holds for generic extension.

Observe that Theorem 1 implies in particular that the set of coboundaries is closed in C^k for $k > k_0$.

We also present versions of this theorem for extensions of subshifts of finite type.

Our results are also true for relative coboundaries. Let $A_0(x) = \int A(x,y)dy$. We say that A is a relative coboundary if

$$A = A_0 + B - B \circ F. \tag{3}$$

Theorem 2. *The results of Theorem 1 are valid also for relative coboundaries.*

2. Preliminaries.

2.1. **Subshifts of finite type.** Here we present some results about subshifts of finite type and their compact extensions. The proofs can be found in [13], Chapters 3 and 8. For a geometric interpretation of the results about the extensions see e.g. [4], Section 2. Let $\mathfrak{a}$ be a finite alphabet, $\mathbf{A}$ be $\text{Card}(\mathfrak{a}) \times \text{Card}(\mathfrak{a})$–matrix whose entries are zeroes and ones. Let $\Sigma = \Sigma_{\mathbf{A}}$ be associated (two-sided) subshift of finite type, that is $\Sigma = \{\{\omega_i\}_{i=-\infty}^{+\infty}$ such that $\omega_i \in \mathfrak{a}$ and $\mathbf{A}_{\omega_i,\omega_{i+1}} = 1\}$. Let σ be a shift $\sigma(\omega)_i = \omega_{i+1}$. Given $\theta < 1$ we consider the metric d_θ on Σ given by $d_\theta(\omega', \omega'') = \theta^j$ where $j = \max(k : \omega'_i = \omega''_i \text{ for } |i| < k)$. Let $C_\theta(\Sigma)$ denote the set of d_θ-Lipschitz functions. Given $\phi \in C_\theta(\Sigma)$ we denote by μ_ϕ the Gibbs measure with potential ϕ, that is

$$h_{\mu_\phi} + \mu_\phi(\phi) = \sup_\mu (h_\mu + \mu(\phi))$$

where the supremum is taken over all σ invariant probability measures. We will use the fact that homologous functions have the same Gibbs measures. Let G be a compact connected Lie group and $\tau \in C_\theta(\Sigma, G)$. Let $\tilde{N} = \Sigma \times Y$. Define $\tilde{F} : \tilde{N} \to \tilde{N}$ by $F(\omega, y) = (\sigma\omega, \tau(x)y)$. Consider a measure ν_ϕ given by $d\nu_\phi = d\mu_\phi dg$. Let $C_{k,\theta}(\tilde{N}) = C_\theta(\Sigma, C^k(G))$. We say that $\tilde{F}$ is rapidly mixing if $\forall\phi\ \forall N\ \exists k$ such that $\forall A, B \in C_{k,\theta}$

$$\left|\nu_\phi(A(\omega,y)B(\tilde{F}^n(\omega,y))) - \nu_\phi(A)\nu_\phi(B)\right| \leq \text{Const}||A||_{k,\theta}||B||_{k,\theta}n^{-N}.$$

It is shown in ([4], Theorem 4.3) that rapid mixing is generic among compact group extensions of subshifts of finite type.

Let Σ^+ be associated one-sided subshift which is defined similarly to Σ but omega is now an one-sided sequence $\omega = \{\omega_i\}_{i=0}^\infty$. $C_\theta(\Sigma^+)$, Gibbs states, rapid mixing etc. are defined for one-sided shifts similarly to two-sided shifts. Let $\tilde{F} : \tilde{N} \to \tilde{N}$ be a skew extension defined by $\tau \in C_\theta(\Sigma, G)$, $\phi \in C_\theta(\Sigma)$ be a potential and $A \in C_{k,\theta}$ be an observable. Then there are $\tau^* \in C_{\sqrt{\theta}}(\Sigma, G)$, $M \in C_{\sqrt{\theta}}(\Sigma, G)$, $\phi^* \in C_\theta(\Sigma)$ $\psi \in C_{\sqrt{\theta}}(\Sigma)$, $A^* \in C_{k,\sqrt{\theta}}(\Sigma^+)$, $K \in C_{k,\sqrt{\theta}}(\Sigma)$ such that

$$\tau^* = (M \circ \sigma)\tau M^{-1}, \quad \phi^* = \phi + \psi - (\psi \circ \sigma), \quad A^* = A + K - K \circ \tilde{F}.$$

Moreover ϕ^* can be chosen so that

$$\forall\omega \quad \sum_{\sigma\varpi=\omega} e^{\phi^*(\varpi)} = 1 \tag{4}$$

Then skew products defined by τ and τ^* are conjugated, ϕ and ϕ^* have the same Gibbs measure and A is a coboundary iff A^* is a coboundary.

Let $\tilde{N}^+ = \Sigma^+ \times Y$ and $\tilde{F}$ be a skew extension defined by some $\tau \in C_\theta(\Sigma^+, G)$. Let Δ be a G–invariant Laplacian on G and let

$$H_\lambda = \{\varphi : \Delta\varphi = \lambda\varphi\}.$$

We endow H_λ with L^2–norm. Denote $C_{\lambda,\theta}(\Sigma^+) = C_\theta(\Sigma^+, H_\lambda)$. Let ϕ be any Hölderfunction on Σ^+ such that

$$\forall\omega \quad \sum_{\sigma\varpi=\omega} e^{\phi(\varpi)} = 1 \tag{5}$$

and let μ_ϕ be the Gibbs measure for ϕ. Consider the transfer operator

$$(\mathcal{L}(h))(\omega, g) = \sum_{\sigma\varpi=\omega} e^{\phi(\varpi)} h(\varpi, \tau^{-1}(\varpi)g). \tag{6}$$

Then $\mathcal{L}$ preserves $C_{\lambda,\theta}(\Sigma^+)$. Let $\mathcal{L}_\lambda$ denote the restriction of $\mathcal{L}$ to $C_{\lambda,\theta}(\Sigma^+)$.

Proposition 1. *([4], Proposition 4.4) If $\tilde{F}$ is rapidly mixing then $\exists C, s$ such that*

$$||\mathcal{L}_\lambda^n|| \leq C\lambda^s \left(1 - \frac{1}{C\lambda^s}\right)^n. \tag{7}$$

2.2. **Anosov diffeomorphisms.** Recall that a diffeomorphism $F : M \to M$ is called Anosov if there is an f–invariant splitting

$$TM = E^s \oplus E^u$$

and constants $C, \rho < 1$ such that

$$\forall v \in E^s \;\; ||df^n v|| \leq C\rho^n ||v||, \quad \forall v \in E^u \;\; ||df^{-n} v|| \leq C\rho^n ||v||.$$

The distributions E^s and E^u are uniquely integrable, they are tangent to foliation W^s and W^u respectively. Since W^s and W^u are transverse, if $x, y \in M$ are close to each other the intersection $W^u_{loc}(x) \bigcap W^s_{loc}(y)$ consists of one point which we denote by $[x, y]$. A set Π is called parallelogram if for all $x, y \in \Pi$ one has $[x, y] \in \Pi$. A partition $\mathbf{\Pi} = \{\Pi_1, \Pi_2, \dots \Pi_n\}$ is called Markov if for all $x \in \mathrm{Int}(\Pi_i)$

$$f W^s_\Pi(x) \in W^s_\Pi(fx), \quad f^{-1} W^u_\Pi(x) \in W^u_\Pi(f^{-1}x)$$

where $W^*_\Pi(z) = W^*_{loc}(z) \bigcap \Pi_j$ if $z \in \Pi_j$. Given a Markov partition $\mathbf{\Pi}$ one can consider a subshift of finite type Σ with $\mathfrak{a} = \{1, 2 \dots n\}$ and $\mathbf{A}_{ij} = 1$ iff $f(\mathrm{Int} P_i) \bigcap P_j = 0$. The map $\zeta : \Sigma \to M$ given by

$$\zeta(\omega) = \bigcap_j f^{-j} \Pi_{\omega_j}$$

defines a semicongugacy between σ and f. If τ is a function from M to G let $\bar{\tau} = \tau\zeta$. Then $\zeta \times \mathrm{id}$ is a semicongugacy between

$$F(x, y) = (fx, \tau(x)y) \quad \text{and} \quad \tilde{F}(\omega, y) = (\sigma\omega, \bar{\tau}(\omega)y).$$

We shall use the fact that the skew extension F is partially hyperbolic. That is, there is an F invariant splitting

$$TN = E^s_F \oplus E^c_F \oplus E^u_F$$

and constants $C, \rho < 1$ such that

$$\forall v \in E^s_F \;\; ||dF^n v|| \leq C\rho^n ||v||, \quad \forall v \in E^u_F \;\; ||dF^{-n} v|| \leq C\rho^n ||v||$$

and E^c_F is the tangent space to the fibers. Gibbs states for f are defined similarly it was done for σ. An important special case is so called SRB measure which is the Gibbs measure with potential

$$\phi_{SRB} = -\ln \det(df | E^u).$$

The importance of SRB measure comes from the fact that if $\Phi \in C(M)$ then

$$\frac{1}{n} \sum_{j=0}^{n-1} \Phi(f^j x) \to \mu_{SRB}(\Phi), \quad n \to +\infty$$

for Lebesgue almost all x.

$\forall\alpha\exists\theta$ such that if $\phi\in C^\alpha(M)$ then $\bar\phi=\phi\circ\zeta\in C_\theta(\Sigma)$. Now μ_ϕ is a Gibbs state for f iff $\forall\Omega\subset\Sigma$

$$\mu_{\bar\phi}(\Omega)=\mu_\phi(\zeta(\Omega)).$$

We say that F is rapidly mixing if $\forall\phi\ \forall N\ \exists k$ such that $\forall A,B\in C^k(M)$

$$|\nu_\phi(A(x,y)B(F^n(x,y)))-\nu_\phi(A)\nu_\phi(B)|\leq \text{Const}||A||_k||B||_k n^{-N}.$$

Then F is rapidly mixing iff $\tilde F$ is rapidly mixing.

3. Symbolic systems.

3.1. **One-sided shifts.** In this subsection let $\tilde F:\tilde N^+\to\tilde N^+$ be a rapidly mixing extension of one sided subshift. Let $C_{r,\theta}(\Sigma^+)=C_\theta(\Sigma^+,C^r(Y))$. We prove the following result.

Lemma 1. *Let $A\in C_{\infty,\theta}(\Sigma^+)$ be an $L^2(\nu_\phi)$ coboundary for some $\phi\in C_\theta(\Sigma^+)$, $A=B-B\circ\tilde F$, where $B\in L^2(\nu_\phi)$. Then B has a version in $C_{\infty,\theta}(\Sigma^+)$. Moreover $\exists k_0$ such that if $A\in C_{k,\theta}(\Sigma^+)$, then $B\in C_{k-k_0,\theta}(\Sigma_+)$ and*

$$||B||_{k-k_0,\theta}\leq \text{Const}(k)||A||_{k,\theta}.$$

Proof. By the discussion of subsection 2.1 we can assume that ϕ satisfies (5). Let $A=A_0+\sum_{\lambda\neq 0}A_\lambda$, where $A_0(\omega)=\int A(\omega,g)dg$, $A_\lambda\in H_\lambda$. Let $B=\sum_\lambda B_\lambda$. Since F commutes with projections to H_λ,

$$A_\lambda=B_\lambda-B_\lambda\circ\tilde F. \tag{8}$$

In particular, $A_0=B_0-B_0\circ\sigma$ and, by [13], $B\in C_\theta(\Sigma^+)$. Hence we can assume without loss of generality that $A_0\equiv 0$. Applying $\mathcal{L}_\lambda$ to (8) we get

$$\mathcal{L}_\lambda A_\lambda=(\mathcal{L}_\lambda-1)\,B_\lambda.$$

Thus $B_\lambda=-(1-\mathcal{L}_\lambda)^{-1}\mathcal{L}_\lambda B_\lambda$. Now there exists $p=p(G)$ such that

$$||A_\lambda||_\lambda\leq\frac{\text{Const}}{\lambda^{k/2-p}}||A||_{k,\theta}. \tag{9}$$

By Proposition 1 there exists s such that

$$||(1-\mathcal{L}_\lambda)^{-1}||\leq\text{Const}\lambda^s. \tag{10}$$

Hence

$$||B_\lambda||_\lambda\leq\text{Const}\lambda^{2s}||A_\lambda||_\lambda\leq\frac{\text{Const}}{\lambda^{k/2-(2s+p)}}\,\|A\|_{k,\theta} \tag{11}$$

Now

$$||B_\lambda||_{k-k_0,\theta}\leq\text{Const}\lambda^{\bar p+\frac{k-k_0}{2}}||B_\lambda||_\lambda. \tag{12}$$

Let $B=\sum_\lambda B_\lambda$. Then
$$||B||_{k-k_0,\theta} \leq \sum_\lambda ||B_\lambda||_{k-k_0,\theta} \leq \text{Const} \sum_\lambda \lambda^{p+\bar{p}+2s-(k_0/2)}||A||_{k,\theta}$$
and this series converges if k_0 is large enough. This completes the proof. □

3.2. **Two-sided shifts.** Let $\tilde{F}:\tilde{N}\to\tilde{N}$ be an extension of two sided subshift of finite type.

Lemma 2. *Let $A=B-B\circ\tilde{F}$. If $A\in C_{\infty,\theta}(\Sigma)$, then $B\in C_{\infty,\theta^{1/4}}(\Sigma)$. Moreover $\exists k_0$ such that if $A\in C_{k,\theta}(\Sigma)$, then $B\in C_{k-k_0,\theta^{1/4}}(\Sigma)$ and*
$$||B||_{k-k_0,\theta^{1/4}} \leq \text{Const}(k)||A||_{k,\theta}.$$

Proof. Let $\tau^*=(M\circ\sigma)\tau M^{-1}$. Then the change of variables $y^*=My$ conjugates $\tilde{F}$ and $F^*(\omega,y^*)=(\sigma\omega,\tau^*(\omega)y^*)$. Thus A is $\tilde{F}$–coboundary iff $A^*(\omega,y^*)=A(\omega,M^{-1}y^*)$ is F^*–coboundary. Write $A^*=A^{**}+K^*-K^*\circ F^*$ where $A^{**}\in C_{k,\theta^{1/4}}(\Sigma^+)$. Then A^* is F^*-coboundary iff A^{**} is. But by Lemma 1 $A^{**}=B^{**}-B^{**}\circ F$ where $B^{**}\in C_{k-k_0,\theta^{1/4}}(\Sigma^+)$. Thus
$$A=(B+K)-(B+k)\circ\tilde{F}$$
where $B(\omega,y)=B^{**}(\omega,y)$, $K(\omega,y)=K^*(\omega,My)$. and the statement follows. □

Corollary 1. *If ω is a periodic orbit of σ, say $\sigma^n\omega=\omega$, then*
$$\left|\sum_{j=0}^{n-1} A_\lambda(\tilde{F}^j(\omega,y))\right| \leq C\lambda^s||A_\lambda||_{\theta,k_0}d(\tau_n(\omega)y,y).$$

Proof.
$$\left|\sum_{j=0}^{n-1} A_\lambda(\tilde{F}^j(\omega,y))\right| \leq |B_\lambda(\omega,\tau_n(\omega)y)-B_\lambda(\omega,y)| \leq \sqrt{\lambda}d(\tau_n(\omega)y,y)||B_\lambda||$$
and the result follows by (11). □

4. Anosov diffeomorphisms.

4.1. **Höldercontinuity.** Now we start a proof of Theorem 1. In this subsection we show that B has a Hölderversion. Let $\mathbf{\Pi}$ be a Markov partition of M and let Σ be the associated subshift of a finite type. Let $\zeta:\Sigma\to M$ be the semiconjugacy
$$\zeta\circ\sigma=f\circ\zeta.$$
Let $\bar{\tau}$ and $\tilde{F}$ be as in subsection 2.2. Define $\bar{A}=A\circ\zeta$. Let $\bar{A}=\bar{B}-\bar{B}\circ\tilde{F}$, $\bar{B}=\sum_\lambda\bar{B}_\lambda$. Let $B_\lambda=\bar{B}_\lambda\circ\zeta^{-1}$, $B=\sum_\lambda B_\lambda$. Since ζ^{-1} is discontinuous

we can not deduce immediately that B_λ are Hölder. Rather we obtain is a consequence of periodic leaves estimates of Corollary 1. Let $p = (x, y)$ have a dense orbit. We have

$$B_\lambda(F^n p) = B_\lambda(p) - \sum_{j=0}^{n-1} A_\lambda(F^j p).$$

Lemma 3. $B_\lambda|_{\mathrm{Orb}(x,y)}$ *is uniformly Höldercontinuous with Hölderconstant* $C||A_\lambda||\lambda^s$.

Proof. Let $m < n$ and

$$d(F^n p, F^m p) \leq \varepsilon.$$

Denote $k = n - m$, $z = f^m x$, $q = F^m p$. By Anosov Closing Lemma $\exists \tilde{x} \in M$ such that $f^k \tilde{x} = \tilde{x}$ and

$$d(f^j z, f^j y) \leq C d(f^k z, z)^\gamma \rho^{\max(j, k-j)}$$

for some $\gamma > 0$ and $\rho < 1$. Let $u = (\tilde{x}, y)$ then

$$\left|B_\lambda(f^k q) - B_\lambda(q)\right| = \left|\sum_{j=0}^{k-1} A_\lambda(F^j q)\right| \leq \left|\sum_{j=0}^{k-1} A_\lambda(F^j u)\right| + \left|\sum_{j=0}^{k-1} \left[A_\lambda(F^j q) - A_\lambda((F^j u)\right]\right|.$$

Now by Corollary 1 the first part is

$$O\left(||A_\lambda||\lambda^{s_4} d^\alpha(p, F^k p)\right)$$

and the second part is

$$O\left(||A_\lambda||\sqrt{\lambda} d^\gamma(p, F^k p)\right)$$

since A_λ is Lipschitz with constant $\sqrt{\lambda}||A_\lambda||$. □

Since $\mathrm{Orb}(p)$ is dense we can extend B_λ to Hölderfunctions on N.

Lemma 4. *Under the conditions of Theorem 1 the restriction of* B *to each fiber is smooth. Moreover* $\exists k_0$ *such that*

$$||B||_{C^{\alpha'}(M, C^{k-k_0}(G))} \leq \mathrm{Const}(k)||A||_{C^\alpha(M, C^k(G))}.$$

Proof. We first show that B has a Hölderversion. By Lemma 3 each B_λ has an extension from $\mathrm{Orb}(x, g)$ to N which is Hölderwith Höldernorm at most $\mathrm{Const}||A_\lambda||\lambda^s$. By continuity of A_λ this extension satisfies $A_\lambda = B_\lambda - B_\lambda \circ F$. Now

$$||A_\lambda|| \leq \frac{\mathrm{Const}}{\lambda^{(k/2)-p}}||A||_{C^k(N)}.$$

Let $B = \sum_\lambda B_\lambda$ then

$$||B||_{C^\alpha(N)} \leq \sum_\lambda ||B_\lambda||_{C^\alpha(N)} \leq \mathrm{Const}\left(\sum_\lambda \lambda^{p+s-k/2}\right)||A||_{C^k(N)}$$

and if k is large enough this series converges. In other words there exists k_1 such that

$$B_{C^\alpha(N)} \leq \text{Const}||A||_{C^{k_1}N}.$$

Applying this to $\Delta^m A$ we get

$$||B||_{C^\alpha(M,\mathcal{H}^{2m}(G))} \leq \text{Const}||\Delta^m B||_{C^\alpha(N)} \leq \text{Const}||\Delta^m A||_{C^{k_1}(N)} \leq$$

$$\text{Const}||A||_{C^{k_1+2m}(N)}$$

and the result follows by Sobolev embedding theorem. □

4.2. **Smoothness.** We now establish the smoothness of B in the transverse directions.

Lemma 5. *Restrictions of B to the leaves of W_F^s, W_F^u are smooth.*

Proof. It is enough to consider W_F^s. We have $A(p) = B(p) - B(Fp)$. Thus $B(p) = A(p) + B(Fp)$. Hence if $p \in W^s(p_0)$ then

$$B(p) - B(p_0) = \sum_{j=0}^{\infty} \left[A(F^p j) - A(F^j p_0)\right].$$

Since F^j are contractions on W_F^S it is clear that this series can be differentiated term by term as many times as we want (see [3]) . □

We now make use of the following fact ([10]).

Proposition 2 (Journe Lemma)**.** *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two continuous transverse foliations with smooth leaves. Let B be a continuous function whose restrictions on leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ are smooth. Then B is smooth. Moreover there exists k_0 such that if restrictions of B to the leaves are C^k then B is C^{k-k_0}.*

This proposition implies in view of Lemmas 4 and 5 that B is smooth on each leave of W^{sc} and since it is also smooth on each leave of W^u we conclude that B is smooth. This complete the proof of Theorem 1.

Remark. *Weaker versions of Journe Lemma proven in* [3, 9] *would also suffice for the proof.*

4.3. **Relative coboundaries.** *Proof of Theorem 2.* Apply Theorem 1 to ΔA. □

4.4. **A counter-example.** Let $G = \mathbb{T}^2$,

$$F(x, t_1, t_2) = (fx, t_1 + \alpha_1 r(x), t_2 + \alpha_2 r(x)).$$

Suppose that α_1/α_2 is irrational and that for all N there exist $m_{1,N}, m_{2,N} \in \mathbb{Z}$ such that

$$|\alpha_1 m_{1,N} + \alpha_2 m_{2,N}| \le m_{2,n}^{-N}.$$

By reindexing we can assume that $m_{2,N} > N^2$. Let $\Phi_N(x, t_1, t_2) = \exp(2\pi i(m_{1,N} t_1 + m_{2,N} t_2))$. Then

$$\Phi \circ F = \exp(2\pi i(m_{1,N}\alpha_1 + m_{2,N}\alpha_2) r(x))\Phi_N.$$

Let $A = \sum_N ((\Phi_N - \Phi \circ F)/N^2)$. Then $A \in C^\infty(N)$, $A = B - B \circ F$ where $B = \sum_N \Phi_N \in C^0(N) - C^1(N)$. By considering suitable linear combinations of Φ_N it is easy to see that F is not rapidly mixing. This shows that arbitrary extensions need not satisfy Theorem 1.

4.5. **Obstructions.** We now provide some criteria for function being a coboundary. Most of these criteria come from other papers, however their applicability is a consequence of the fact that different notions of coboundaries coincide in our situation. Sometimes it is easier to verify that A is a relative coboundary. It is also perfectly satisfactory since it is well known when A_0 is an f-coboundary.

(i) Define

$$\mathcal{D}_\phi(A) = \nu_\phi(A^2) - \nu_\phi^2(A) + 2\sum_{j=1}^{\infty}\left[\nu_\phi(A(A \circ \tilde{F}^j))\nu_\phi^2(A)\right].$$

Proposition 3. *([6]) A is a cohomologous to a constant $\Leftrightarrow \exists\phi$ such that $\mathcal{D}_\phi(A) = 0 \Leftrightarrow \forall\psi$ $\mathcal{D}_\phi(A) = 0$.*

Proof. Without loss of generality we can assume that $\nu_\phi(A) = 0$. Then $D_\phi(A) = 0 \Leftrightarrow A$ is $L^2(\nu_\phi)$-coboundary (Spectral theorem)
$\Leftrightarrow A$ is Höldercoboundary (Theorem 1)
$\Rightarrow \forall\psi \quad A$ is $L^2(\nu_\psi)$-coboundary. □

(ii) Let $P = \{p_0, p_1 \ldots p_n\}$ be a chain such that $p_{k+1} \in W^s(p_k) \bigcup W^u(p_k)$. We say that P is closed if $p_0 = p_n$. Define

$$r(P) = \sum_k P(p_k, p_{k+1})$$

where

$$r(p_k, p_{k+1}) = \begin{cases} \sum_{j=0}^{\infty}[A(F^j p_{k+1}) - A(F^j p_k)] & \text{if } p_{k+1} \in W^s p_k \\ \sum_{j=-\infty}^{-1}[A(F^j p_k) - A(F^j p_{k+1})] & \text{if } p_{k+1} \in W^s p_k \end{cases}$$

The following statement is Corollary 3.1 from [11].

Proposition 4. *If F has accessibility property then A is cohomologous to a constant if and only if for any closed chain P, $r(P) = 0$.*

(iii) The next result is clear from the proof of Theorem 1.

Proposition 5. *A is a relative coboundary $A \Leftrightarrow \forall N \in \mathbb{N}\ \ \Delta^N A$ is a coboundary.*

(iv) Let $G = \mathbb{T}$. Let ν be the SRB measure for F. Consider a one parameter family

$$F_\varepsilon(x, z) = (fx, z + \tau(x) + \varepsilon A(x, z) + \varepsilon^2 \alpha(\varepsilon, x, z).$$

Let ν_ε be any u-Gibbs measure for F_ε, that is, the projection of ν_ε to M is μ_{SRB}.

Proposition 6. *A is a relative coboundary if and only if*

$$\lambda_c(\nu_\varepsilon) = o(\varepsilon^2).$$

Proof. We use the following asymptotics ([5])

$$\nu_\varepsilon(H) = \nu(H) + \varepsilon\omega(H) + o(\varepsilon)$$

where

$$\omega(H) = \sum_{j=1}^{\infty} \nu \left(A \circ F^{-j} \frac{dH}{dz} \right).$$

We want to apply this to $H_\varepsilon = \ln \frac{dF_\varepsilon}{dz}$. We have

$$\frac{dF_\varepsilon}{dz} = 1 + \varepsilon \frac{dA}{dz} + \varepsilon^2 \frac{d\alpha(0, x, z)}{dz} + o(\varepsilon^2).$$

So

$$\ln \frac{dF_\varepsilon}{dz} = \varepsilon \frac{dA}{dz} + \varepsilon^2 \left[\frac{d\alpha(0, x, z)}{dz} - \frac{1}{2} \left(\frac{dA}{dz} \right)^2 \right] + o(\varepsilon^2).$$

Hence

$$\nu_\varepsilon \left(\ln \frac{dF_\varepsilon}{dz} \right) =$$

$$\varepsilon\nu \left(\frac{dA}{dz} \right) + \varepsilon^2 \left[\nu(\frac{d\alpha(0, x, z)}{dz}) - \frac{1}{2} \nu \left(\left(\frac{dA}{dz} \right)^2 \right) + \sum_{j=1}^{\infty} \nu \left(A \circ F^{-j} \frac{d^2 A}{dz^2} \right) \right] + o(\varepsilon^2).$$

Now since $d\nu = d\mu_{SRB} dg$ it follows that

$$\nu \left(\frac{dA}{dz} \right) = \nu \left(\frac{d\alpha(0, x, z)}{dz} \right) = 0 \quad \text{and}$$

$$\nu \left(A \circ f^{-j} \frac{d^2 A}{dz^2} \right) = -\nu \left(\left(\frac{dA}{dz} \right) \circ F^{-j} \frac{dA}{dz} \right).$$

Hence

$$\nu_\varepsilon(\ln \frac{dF_\varepsilon}{dz}) \sim -\varepsilon^2 \left[\frac{1}{2}\nu\left(\left(\frac{dA}{dz}\right)^2\right) - \sum_{j=1}^{\infty} \nu\left(\left(\frac{dA}{dz}\right)\circ F^{-j}\frac{dA}{dz}\right)\right]$$

That is

$$\nu_\varepsilon(\ln \frac{dF_\varepsilon}{dz}) \sim -\frac{\varepsilon^2 \mathcal{D}_{SRB}(\frac{dA}{dz})}{2}. \tag{13}$$

Therefore

$$\lambda_c(\nu_\varepsilon) = o(\varepsilon^2)$$

if and only if $\mathcal{D}_{SRB}(\frac{dA}{dz}) = 0$. By Proposition 3 $\frac{dA}{dz}$ is a coboundary that is A satisfies (3). □

Appendix A. Non-mixing case.

Observe that Anosov times rotation could satisfy the conclusions of Theorem 1 even though it is not mixing. In this appendix we give an extension of Theorem 1 to a non-mixing case. In order to explain our result let us recall some background. Given τ, let $\bar{N} = M \times G$ and consider the principal extension $\bar{F} : \bar{N} \to \bar{N}$ given by $\bar{F}(x, g) = (f(x), \tau(x)g)$. Recall the definition of Brin groups [1, 2]. Given a partially hyperbolic diffeomorphism we call a sequence $P = \{p_1, p_2 \dots p_n\}$ a e-chain (respectively t-chain) if $p_{j+1} \in W^u(p_j) \bigcup W^s(p_j)$ (respectively $p_{j+1} \in W^u(p_j) \bigcup W^s(p_j) \bigcup \mathrm{Orb}(p_j)$. Fix a reference point $x \in M$. Given any chain $P \subset M$ with $x_n = x_1 = x$ and any $g_1 \in G$ there is unique chain $\bar{P} \subset \bar{N}$ starting at (x, g_1) and covering P. $\bar{P}$ is not closed, rather $g_n = g(P)g_1$. Let $\Gamma_t(x)$ ($\Gamma_e(x)$) denote the set of all $g(P)$ for all closed t-chains (respectively e-chains) starting at x.

Proposition 7. *(Brin) ([1, 2]) (a) $\Gamma_*(x)$ are groups. Γ_* of different points are conjugated, Γ_t is a normal subgroup of Γ_e, Γ_e/Γ_t is cyclic. In particular $\bar{\Gamma}_e/\bar{\Gamma}_t$ is abelian.*

(b) (F, ν_ϕ) is ergodic$\Leftrightarrow$ $\bar{\Gamma}_e$ acts transitively on Y.

(F, ν_ϕ) is mixing$\Leftrightarrow$ $\bar{\Gamma}_t$ acts transitively on Y.

A quantitative version of this result was obtained in [4]. Call a set $S \subset G$ Diophantine on Y if there are constants K, σ such that for any function h on Y with $\Delta h = \lambda h$ then there is $s \in S$ such that

$$||h - h \circ s|| \leq \frac{K}{\lambda^\sigma}||h||_{L^2}.$$

Let $\Gamma_t(x, R)$ ($\Gamma_e(x, R)$) denote the set of $g(P)$ for all chains $P = (x_1, x_2 \dots x_n)$ with $x_1 = x_n = x$, $n \leq R$, $d_{W^*}(x_j, x_{j+1}) \leq R$ (if $x_{j+1} = f^m x_j$ we require that $|m| \leq R$).

Proposition 8. *([4] (a) S is Diophantine on Y iff S is Diophantine on $Y/[G,G]$ and $Y/\text{Center}(G)$.*

(b) S is Diophantine on $Y/\text{Center}(G)$ $\Leftrightarrow$ there are no S-invariant functions$\Leftrightarrow$ S contains a finite Diophantine subset.

(c) F is rapidly mixing $\Leftrightarrow$ $\Gamma_t(R)$ is Diophantine for large R.

It is proven in [1] that there is an open and dense subset of pairs (f,τ) such that $\Gamma_t(R)=G$ for large R. The goal of this appendix is to prove the following statement.

Theorem 3. *Suppose that F is ergodic. If $\Gamma_e(R)$ is Diophantine for large R then any solution to (1) satisfies the tame estimates (2).*

Remark. *I believe that the above condition is also necessary for (2) but the approach of subsection 4.4 (cf. also [4], subsection 4.3) shows only that if $\Gamma_e(R)$ is not Diophantine for large R and $A=B-B\circ F$ then the norm of $\partial_y^\alpha B$ can not be bounded by norms of $\partial_y^\beta A$. It does not rule out the possibility that it can be bounded by norms of $\partial_y^{\beta_1}\partial_x^{\beta_2}A$, even though this seems unlikely.*

Proof. Observe that the only place where we have used rapid mixing (i.e. Diophantineness of $\Gamma_t(R)$) was (10). Hence we need to show that (10) holds under a weaker condition that $\Gamma_e(R)$ is Diophantine. To this end we estimate $(1-\mathcal{L}_\lambda)^{-1}$ using the series

$$(1-\mathcal{L}_\lambda)^{-1}=\frac{1}{2}\left(1-\frac{1+\mathcal{L}_\lambda}{2}\right)^{-1}=\frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{1+\mathcal{L}_\lambda}{2}\right)^{j}.$$

Thus instead of Proposition 1 we need to show that there exist C,s such that

$$\left\|\left(\frac{1+\mathcal{L}_\lambda}{2}\right)^{n}\right\|\leq C\lambda^s\left(1-\frac{1}{C\lambda^s}\right)^{n} \tag{14}$$

The proof of (14) is similar to the proof of (7) which is Proposition 4.4 of [4]. Let us describe the modifications needed. Repeating the arguments on page 184 of [4] we find that if (14) fails then for each C_1,β_4 there exist λ,H such that $||H||_{C^0}\leq 1$, $L(H)\leq \text{Const}\lambda$ and $||(\frac{1+\mathcal{L}_\lambda}{2})^{m(\lambda)}H||\geq 1-|\lambda|^{-\beta_4}$ where $m(\lambda)=C_1\ln\lambda$ and $L(H)$ denotes the Lipschitz norm $H:\Sigma^+\to L^2(Y)$. As in [4] this implies that $\forall\tilde\omega,\hat\omega$

$$||\pi_\lambda(\tau_m(\tilde\omega))H(\tilde\omega)-\pi_\lambda(\tau_m(\hat\omega))H(\hat\omega)||\leq\lambda^{-\beta_5}$$

where $\beta_5\to\infty$ as $\beta_4\to\infty$. However in the present setting we also have that for all ω

$$||\pi_\lambda(\tau(\omega))H(\omega)-H(\sigma\omega)||\leq\lambda^{-\beta_5} \tag{15}$$

Indeed the expression for $[(\frac{1+\mathcal{L}_\lambda}{2})^m H](\sigma\omega)$ contains among various terms

$$(1/2)^m[H(\sigma\omega) + e^{\phi(\omega)}\pi_\lambda(\tau(\omega))H(\omega)].$$

These two vectors should be almost collinear in the sense of [4], page 185 proving (15).

(15) implies that in our setting Lemma 4.7 of [4] holds for e-chains and not only for t-chains as in [4]. Continuing as in [4], page 186 we show that if (14) fails then this contradicts to Diophantiness of Γ_e. Thus (14) holds. Thus Theorem 1 holds under the assumption that $\Gamma_e(R)$ is Diophantine as claimed. □

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