RECURRENCE PROPERTIES OF PLANAR LORENTZ PROCESS

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1. Introduction

The (periodic) Lorentz process (PLP) is the $\mathbb{Z}^d-$covering of a Sinai billiard, in other words of a dispersing billiard, given on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. If the horizon is finite, i. e. the free flight vector $\kappa(x)$ is bounded, then the Lorentz process is a most instructive model of the Brownian Motion. Indeed, for $d = 2$ and in the diffusive scaling, this (mechanical) process converges weekly to the Wiener process ([BS 81] and [BCS 91]) in the same way as a gem of classical probability theory, the (stochastic) simple symmetric random walk (SSRW) does.

It is natural to expect that more refined properties of the SSRW also hold for the PLP. Nice examples are the local central limit theorem ([SzV 04]), Pólya's recurrence theorem ([Sch 98], [Con 99], [SzV 04]) and the law of iterated logarithm ([Ch 06]).

The main aim of this paper is the study of further delicate probabilistic properties of the PLP. The results are interesting not in themselves, only, but can also be used for treating the locally perturbed Lorentz process ([DSV 06]).

For simplicity, let us consider a Sinai billiard on $\mathbb{T}^2$ with a finite number of disjoint, strictly convex, $\mathcal{C}^3$ scatterers and with finite horizon. For our study it will be more convenient to work with the discrete-time mapping (the Poincaré section of the billiard flow). Denote its phase space by Ω_0, the discrete-time mapping $\Omega_0 \to \Omega_0$ by f_0, and finally the Liouville-measure on Ω_0 by μ_0. Let then Ω denote the phase space of the (infinite) Lorentz-process, i. e. the $\mathbb{Z}^2-$covering of Ω_0 (of course, $\Omega_0 = \Omega/\mathbb{Z}^2$). The analogous objects for Ω are denoted by μ and f. There is a natural projection $\pi : (\Omega, f, \mu) \to (\Omega_0, f_0, \mu_0)$. We can also think of (Ω_0, μ_0) as embedded into (Ω, μ).

Of course, $\Omega_{(0)} = Q_{(0)} \times S_+$ (here the subscript $_{(0)}$ means that the objects in question are defined and the corresponding claims are true for both Ω and Ω_0) where $Q_{(0)}$ denotes the configuration component (i. e. the billiard table) and S_+ is the space (semicircle) of outgoing velocities. The natural projection $\Omega_{(0)} \to Q_{(0)}$ will be denoted by π_q.

Ω_0 is, in fact, the fundamental domain of the configuration space of the Lorentz process and further we denote $Q_m = Q_0 + m \quad (m \in \mathbb{Z}^2)$. Then the meaning of the shifted billiard phase space Ω_m and of the Liouville measure μ_m living on Ω_m; $m \in \mathbb{Z}^2$ is clear. Let S_n be the location (i. e. the configuration component) of the Lorentz particle after n collisions. More formally: let us define the free flight vector $\kappa : \Omega_{(0)} \to \mathbb{R}^2$ as follows: for $x \in \Omega$ let $\kappa(x) = \pi_q(f(x)) - \pi_q(x)$ and for $x \in \Omega_0$ let $\kappa(x) = \kappa(\pi^{-1}(x))$. Then

$$S_n(x) = \sum_{k=0}^{n-1} \kappa(f_{(0)}^k(x)).$$

Let $m(S) = m$ if $S \in Q_m$. Let

$$\tau = \min\{n > 0 : m(S_n) = 0\} \quad (i.\ e.\ \tau : \Omega \to \mathbb{N}) \tag{1}$$

In this paper we prove the following results.

Theorem 1. *There is a constant* $\mathbf{c}$ *such that* $\mu_0(\tau > n) \sim \frac{\mathbf{c}}{\log n}$.

Theorem 2. *Let* $N_n(x) = \mathrm{Card}(k \leq n : m(S_k) = 0)$. *Then if* x *is distributed according to* μ_0, *then* $\mathbf{c}N_n/\log n$ *converges to a mean 1 exponential distribution.*

Combining this with Hurewicz's Ratio Ergodic Theorem (see [Pet 83], Section 3.8) we obtain the following

Corollary 3. *If* $A \in L^1(\mu)$, *then for almost every* x *the sum* $\sum_{j=0}^{n-1} A(f^j x)/\log n$ *converges to an exponential random variable with mean* $\mu(A)/\mathbf{c}$.

Let t_m denote the random variable $\tau(x)$ under the condition that x is distributed according to μ_m.

Theorem 4. *As* $|m| \to \infty$, $\log t_m/2\log|m|$, *converges weakly to a random variable* ξ *where*

$$\mathbb{P}(\xi > r) = \frac{1}{\max(r, 1)}. \tag{2}$$

Let ν_m denote the distribution of $f^\tau(x)$ if x is distributed according to μ_m.

Theorem 5. *As* $|m| \to \infty$, ν_m *converges to a limiting measure* ν.

Even though our results are formulated for the Poincaré map they can be used to obtain information about continuous time dynamics.

Let $(\mathcal{M}_0, g_0^t)$ be the flow of the associated Sinai billiard and $(\mathcal{M}, g^t)$ be the Lorentz flow (in other words, their Poincaré section dynamics are (Ω_0, f_0, μ_0) and (Ω, f, μ) respectively). Let $\mathbf{m}_0$ and $\mathbf{m}$ denote the corresponding Lioville measures. By a natural identification the time T_n of the n-th collision coincides for these flows. Let $\bar{\mathcal{M}}$ be a fundamental domain for $\mathcal{M}$ and $\bar{\mathcal{M}}_m$ denote the domain which is obtained from $\bar{\mathcal{M}}$ by shifting positions of all points by m.

Let $\bar{L} = \text{Area}(Q_{(0)})/l$ denote the mean free path where l is the total length of the scatterers in $Q_{(0)}$.

Corollary 3 and Theorem 4 imply the following about the continuous time dynamics.

Corollary 6. *If $A \in L^1(\mathbf{m})$, then for almost every x the integral $\int_0^t A(g^s x)ds/\log t$ converges to an exponential random variable with mean $\bar{L}\mathbf{m}(A)/\mathbf{c}$.*

Corollary 7. *Let $\mathbf{t}$ be the first continuous time when $g^t x \in \bar{\mathcal{M}}$ and denote by $\mathbf{t}_m$ the random variable $\mathbf{t}(x)$ where x is uniformly distributed on $\bar{\mathcal{M}}_m$. As $|m| \to \infty$, $\log \mathbf{t}_m/(2\log|m|)$, converges weakly to a random variable ξ whose distribution is given by* (2).

2. Extensions

By linear Lorentz process (LLP) we mean a particle moving in a periodic configuration of scatterers at either strip or cylinder. Again we assume finite horizon. Theorems of the previous section have natural generalizations to LLP.

For the next two results we assume that x_0 is distributed according to μ_0. Let

$$\tau^* = \min\{k : S_k \in O\}.$$

Theorem 8. *There is a constant $\bar{\mathbf{c}}$ such that $\mathbb{P}(\tau^* > n) \sim \frac{\bar{\mathbf{c}}}{\sqrt{n}}$.*

Theorem 9. *Let $N_n(x) = \text{Card}(k \leq n : m(S_k) = 0)$. Then $\mathbf{c}N_n/\sqrt{n}$ converges to Mittag-Leffler distribution of index* $1/2$.

Recall that Mittag-Leffler distribution Y of index $1/2$ has moments

$$\mathbb{E}(Y^k) = c^k \frac{k!}{\Gamma(\frac{k}{2}+1)}.$$

Fix a scatterer O. For the next two results we assume that x_0 is uniformly distributed on $\pi_q^{-1} O_m$ where O_m denotes O shifted by m and that $|m| \gg 1$. Let τ be the first time when $S_m \in O$. Let τ_m be the distribution of τ.

Theorem 10. *There is a constant σ^2 such that $\frac{\sigma^2 \tau_m}{|m|^2}$ converges weakly to $\mathbf{t}$–the first time then the standard Brownian motion visits* 1.

Theorem 11. *The distribution of x_{τ_m} approaches limits as $m \to \pm\infty$.*

Our techniques can also be applied in other settings. For example, let x', x'' be two independent Lorentz particles. Suppose that, at time 0, x' is uniformly distributed on $\bar{\mathcal{M}}_0$ and x'' is uniformly distributed on $\bar{\mathcal{M}}_m$ Let $\tau(x', x'')$ be the first time when $d(g_t(x'), g_t(x'')) \leq 1$ (i. e. $\tau(x', x'') : \bar{\mathcal{M}}_0 \times \bar{\mathcal{M}}_m \to \mathbb{N}$). The proofs of the following theorems are similar to the proofs of Theorems 4 and 5.

Theorem 12. *As $|m| \to \infty$, $\log \tau(x', x'')/2 \log |m|$ converges weakly to a random variable ξ whose distribution is given by* (2).

Theorem 13. *As $|m| \to \infty$, the distribution of the vector $(\pi(g_\tau(x')), \pi(g_\tau(x'')))$ converges to a limiting one.*

3. Preliminaries

In this section notions and theorems are collected, which later will be used or referred to.

3.1. **Hyperbolicity of the billiard map.** For definiteness, let $Q_0 = \mathbb{T}^2 \setminus \cup_{i=1}^p O_i$ where the closed sets O_i are pairwise disjoint, strictly convex with $\mathcal{C}^3$−smooth boundaries. In $\Omega_{(0)}$ it is convenient to use the product coordinates which, for simplicity, we only introduce for Ω_0. Recall that

$$\Omega_0 = \{x = (q, v) | q \in Q_0, \langle v, n \rangle \geq 0\}$$

where $\langle \cdot, \cdot \rangle$ denotes scalar product, and n is the outer normal in the collision point. Traditionally for q one uses the arclength parameter and for the velocity the angle $\phi = \arccos \langle v, n \rangle \in [-\pi/2, \pi/2]$. In these coordinates the invariant measure is given by the density $\frac{1}{2l} \cos \phi \, dq \, d\phi$, where l is the overall perimeter of the scatterers. From our assumptions it follows that $0 < \min |\kappa| < \max |\kappa| < \infty$.

For our billiards there is a natural Df_0-invariant field $\mathcal{C}_x^u$ of unstable cones (and dually also a field $\mathcal{C}_x^s$ of stable ones) of the form $c_1 \leq \frac{d\phi}{dq} \leq c_2$ (or $-c_2 \leq \frac{d\phi}{dq} \leq -c_1$ respectively) where $0 < c_1 < c_2$ are suitable constants.

A connected smooth curve $\gamma \subset \Omega_0$ is called an *unstable curve* (or a *stable curve*) if at every point $x \in \gamma$ the tangent space $\mathcal{T}_x\gamma$ belongs to the unstable cone $\mathcal{C}_x^u$ (or the stable cone $\mathcal{C}_x^s$ respectively).

For an unstable curve γ (or a stable one) and for any $x \in \gamma$ denote by $\mathcal{J}_\gamma f_0^n(x) = ||D_x f_0^n(dx)||/||dx||$, $dx \in \mathcal{T}_x\gamma$ the *Jacobian* of the map f_0^n at the point x. Then the *hyperbolicity of the dynamics* means that there are constants $\Lambda > 1$ and $C > 0$ depending on the dynamics, only, such that for any unstable (or stable) curve γ and every $x \in \gamma$ and every $n \geq 1$ one has $\mathcal{J}_\gamma f_0^n(x) \geq C\Lambda^n$ (or $\mathcal{J}_\gamma f_0^{-n}(x) \geq C\Lambda^n$ respectively).

Sinai billiards are hyperbolic and, consequently, the Lyapunov exponents are non-zero. Since Sinai's celebrated paper [Sin 70] one knows that much more is true: the billiard is ergodic, K-mixing and has further nice and strong properties. This theory is already standard and for further results and methods it suffices to refer to [Sz 00]. In various recent works, however, new and very efficient non-traditional tools were developed, which will also be used in this work. Though we can not give a detailed exposition, we briefly describe the most important statements in the form we will use them. For more details we refer to the original works.

3.2. **Standard pairs.** Let us start with a heuristic introduction. Sinai's classical billiard philosophy ([Sin 70] reacts to the fact that dispersing billiards are hyperbolic (a nice property) but at the same time they are singular dynamical systems (an unpleasant property). Nevertheless smooth pieces of unstable (and of stable) invariant manifolds do exist for *expansion prevails partitioning.*

Though dispersing billiards are hyperbolic, they are not only singular but, added to that, close to the singularities the derivative of the map also explodes. This circumstance is the most unpleasant when one aims at proving the distortion estimates, basic for the techniques. To cope with this difficulty [BCS 91] introduced the idea of surrounding the singularities with a countable number of extremely narrow, so-called *homogeneity strips*, roughly parallel to the singularities. In these strips the derivative of the map can be large, but oscillates very little; this fact makes it possible to nevertheless establish the necessary distortion estimates. The boundaries of these homogeneity strips provide further singularities (causing further partitioning), the so-called *secondary* ones in contrast to the *primary singularities* (in our case only tangencies). The definition of homogeneity strips depends on a parameter denoted usually k_0. The larger k_0 is, the smaller the neighborhood of (primary) singularities is where one introduces the homogeneity strips. In certain bounds (e. g. in the growth lemmas) k_0 should be selected sufficiently large.

Let us now give precise definitions. For $k \geq k_0$ let

$$\mathbb{H}_k = \{(r, \phi) : \frac{\pi}{2} - k^{-2} < \phi < \frac{\pi}{2} - (k+1)^{-2}\},$$

$$\mathbb{H}_{-k} = \{(r, \phi) : \frac{\pi}{2} - k^{-2} < -\phi < \frac{\pi}{2} - (k+1)^{-2}\},$$

$$\mathbb{H}_0 = \{(r, \phi) : -(\frac{\pi}{2} - k_0^{-2}) < \phi < \frac{\pi}{2} - k_0^{-2}\}.$$

Take $L_1, L_2 \gg 1$ and $\theta < 1$ sufficiently close to 1.

An unstable curve is *weakly homogeneous* if it does not intersect any singularity (i. e. neither primary nor secondary one).

A weakly homogeneous unstable curve γ is *homogeneous* if it satisfies the distortion bound

$$\frac{\log J_\gamma f_0(x)}{\log J_\gamma f_0(y)} \leq L_1 \frac{d(x,y)}{\text{length}^{2/3}(\gamma)} \qquad x, y \in \gamma$$

and the curvature bound

$$\angle(\dot\gamma(x), \dot\gamma(y)) \leq L_1 \frac{d(x,y)}{\text{length}^{2/3}(\gamma)} \qquad x, y \in \gamma$$

We observe that if the $\mathcal{C}^2$ norm of γ is bounded and γ is *long* in the sense that either $\text{length}(\gamma) > \delta_0$ for some fixed constant δ_0 or γ crosses a whole homogeneity strip, then γ satisfies both the distortion and the curvature bounds.

Let $s^+(x,y)$ be the first time $f_0^s(x)$ and $f_0^s(y)$ are separated by a singularity.

A probability density ρ on a homogeneous unstable curve γ is called *a homogeneous density* if it satisfies the density bound

$$|\log \rho(x) - \log \rho(y)| \leq L_2 \theta^{s^+(x,y)}.$$

We will call the connected homogeneous components of an unstable (stable) curve the *H-components* of the curve. Given γ we let $\gamma_n(x)$ be the largest subcurve of $f_0^n \gamma$ containing $f_0^n x$ and such that $f_0^{-n}\gamma_n(x)$ does not contain singularities of f_0^n.

A *standard pair* is a pair $\ell = (\gamma, \rho)$ where γ is a homogeneous curve and ρ is a homogeneous density on γ.

Given a standard pair and a measurable $A : \Omega_0 \to \mathbb{R}$ we write

$$\mathbb{E}_\ell(A) = \int_\gamma A(x) dx$$

and $\text{length}(\ell) = \text{length}(\gamma)$.

In this work the precise definition of the standard pairs is not important but we shall take advantage of their invariance and equidistribution properties listed below and in subsection 3.5.

The fundamental tool used in our work is the so-called *growth lemma.* While hyperbolicity of Sinai billiards means that infinitesimal trajectories diverge exponentially fast, the growth lemma says that the exponential divergence also holds for most trajectories which are sufficiently close to each other.

We give two formulations of the growth lemma. The first and more classical one deals with curves while the second formulation deals with standard pairs.

Let γ be a homogeneous curve and for $n \geq 1$ and $x \in \gamma$ let $r_n(x)$ denote the distance of the point $f_0^n(x)$ from the nearest boundary point of the H-component $\gamma_n(x)$ containing $f_0^n(x)$.

Proposition 3.1. *(Growth lemma). If k_0 is sufficiently large, then*

(a) *there are constants $\beta_1 \in (0,1)$ and $\beta_2 > 0$ such that for any $\varepsilon > 0$ and any $n \geq 1$*

$$\mathrm{mes}_\ell(x:\ r_n(x) < \varepsilon) \leq (\beta_1 \Lambda)^n \mathrm{mes}(x:\ r_0 < \varepsilon/\Lambda^n) + \beta_2 \varepsilon$$

(b) *there are constants $\beta_3, \beta_4 > 0$, such that if $n \geq \beta_3 |\log \mathrm{length}(\gamma)|$, then for any $\varepsilon > 0$ and any $n \geq 1$ one has*

$$\mathrm{mes}_\ell(x:\ r_n(x) < \varepsilon) \leq \beta_4 \varepsilon$$

(c) *If $\ell = (\gamma, \rho)$ is a standard pair, then*

$$\mathbb{E}_\ell(A \circ f_0^n) = \sum_\alpha c_{\alpha n} \mathbb{E}_{\ell_{\alpha n}}(A)$$

where $c_{\alpha n} > 0$, $\sum_\alpha c_{\alpha n} = 1$ and $\ell_{\alpha n} = (\gamma_{\alpha n}, \rho_{\alpha n})$ are standard pairs where $\gamma_{\alpha n} = \gamma_n(x_\alpha)$ for some $x_\alpha \in \gamma$ and $\rho_{\alpha n}$ is the push-forward of ρ up to a multiplicative factor.

(d) *If $n \geq \beta_3 |\log \mathrm{length}(\ell)|$, then*

$$\sum_{\mathrm{length}(\ell_{\alpha n}) < \varepsilon} c_{\alpha n} \leq \beta_4 \varepsilon.$$

Parts (a) and (b) are due to [Ch 99]. The restatement in terms of the standard pairs is taken from [CD 05].

In order to apply standard pairs to the problem at hand observe that the Liouville measure can be decomposed as follows

$$\mu_0(A) = \int \mathbb{E}_{\ell_\alpha}(A) d\sigma(A) \tag{3}$$

where σ is a factor measure such that $\sigma(\mathrm{length}(\ell_\alpha) < \varepsilon) < \mathrm{Const}\varepsilon$.

3.3. **Young towers and transfer operators.** According to our recent understanding the most efficient way for constructing Markov partitions for billiards is to use Young towers, cf. [You98]. We are going to introduce the main concepts without giving a full description.

The presence of singularities prevent stable and unstable curves to admit a lower bound for their size in any part of the phase space. Therefore the product structure which is the key ingredient of several hyperbolic arguments can only be introduced on a complicated set.

First choose an unstable curve W, which is short enough to ensure that a high amount of the points admit unstable curve of this length.

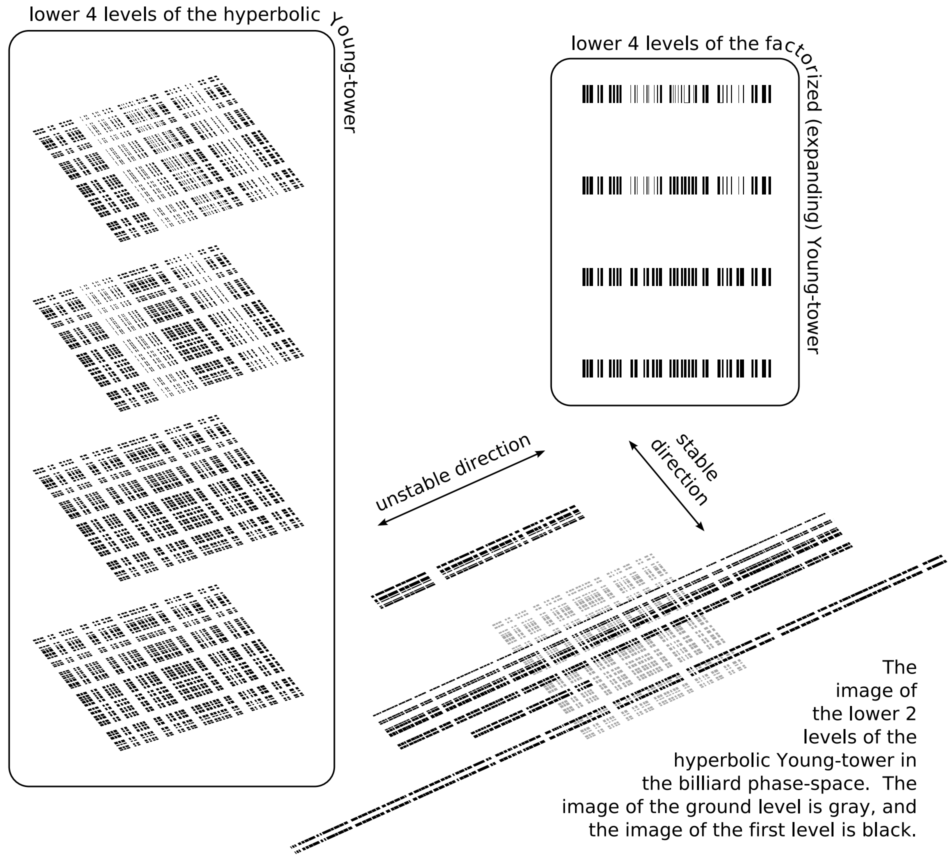

FIGURE 1. Young-towers, and Markov-return

Then define a subset of this curve consisting of points, which remain a certain (exponentially shrinking) distance apart the singularities.

$$\Omega_\infty := \{y \in W \mid d(T^n y, \mathcal{S}) > \delta_1 \lambda^{-n} \quad \forall n \geq 0\}$$

where λ is the hyperbolicity constant. If δ_1 is chosen small enough this set has positive measure. By construction each point in Ω_∞ admits a stable curve of length δ_1.

So far we have one unstable curve W, and a family of stable curves $\{\gamma^s\}$. Let us consider all the nearby unstable curves, which are long enough, and intersect all the stable curves in the previous family. These two families of curves $\{\gamma^s\}$ and $\{\gamma^u\}$ define the hyperbolic product set $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$.

This set is the base of the hyperbolic Young tower. To continue the construction of the tower we are going to focus on recurring subsets of Λ. On figure 1 we can see that some parts of Λ are mapped to Λ. However we are only interested in those returns, which respect the product structure. A subset of Λ is said to be an *s-subset* if it is the product of the full family of stable curves and some part of the unstable family. The notion of *u-subset* is defined *mutatis mutandis* in the same way.

A Markov return is an event when some $f_0^n\Lambda \cap \Lambda$ is a u-subset, and it's inverse image under f_0^{-n} is an s-subset. The possible non-Markov returns are when the intersection is not a u-subset (this is printed as the middle intersection), or when the inverse image is not an s-subset. This latter event happens when a recurring part goes over the edge of Λ in the stable direction.

The inverse image of the Markov recurring part is not necessarily a solid rectangle intersected with Λ. It can have infinitely many "holes" in it, as demonstrated on figure 1.

The tower is built using these Markov type returns. The basic set Λ is divided into s-subsets according to Markov returns, and each subset is marked by the return time R. In this way R will be a function on Λ which is constant on these s-subsets. Not all Markov returns are considered, for sophisticated details please consult [You98]. The tower itself

$$\Delta \stackrel{\text{def}}{=} \{(x,l) : x \in \Lambda;\ \ l = 0, 1, \dots, R(x) - 1\}$$

and the dynamics on the tower is

$$F(x,l) = \begin{cases} (x, l+1) & \text{if} \quad l+1 < R(x) \\ (f_0^R x, 0) & \text{if} \quad l+1 = R(x) \end{cases}$$

Note that we have a decomposition into s-subsets which give rise to a Markov partition on the tower. This tower is hyperbolic, and as a usual tool in this field Young has also introduced a factorized version of it $\bar{\Delta}$. Simply collapse the stable direction. This is also demonstrated on figure 1. We have the following commutative diagram of measure preserving transformations:

$$\begin{array}{ccccc} (\bar{\Delta}, \bar{\mu}_\Delta) & \xleftarrow{\pi_{\bar{\Delta}}} & (\Delta, \mu_\Delta) & \xrightarrow{\pi_{\Omega_0}} & (\Omega_0, \mu_0) \\ {\scriptstyle \bar{F}}\uparrow & & {\scriptstyle F}\uparrow & & {\scriptstyle f_0}\uparrow \\ (\bar{\Delta}, \bar{\mu}_\Delta) & \xleftarrow{\pi_{\bar{\Delta}}} & (\Delta, \mu_\Delta) & \xrightarrow{\pi_{\Omega_0}} & (\Omega_0, \mu_0) \end{array} \qquad (4)$$

The projection to the original phase-space is not 1-1. On figure 1 the intersection in the middle has at least two inverse images. One of them is in the ground floor, and the other is on the first floor. Since the return is not Markovian these point are to be considered as different points on the tower.

Functions on the original phase space Ω_0 can be lifted to Δ. Functions on Δ which are constant along stable directions can be considered as functions on $\bar{\Delta}$. For any function ψ on Δ there exists functions h and φ, such that $\varphi - \psi = h - h \circ F$, and φ is constant along stable directions. In this equation the regularity of the functions can be examined, but we will skip the details, and only introduce distance, and function norms on the factorised tower $\bar{\Delta}$.

The factorised tower $\bar{\Delta}$ has a Cantor structure, and a Markov partition. The Cantor hierarchy can be redefined with the separation time $s(x,y) = \min\{k \geq 0 \mid \bar{F}^k x$ and $\bar{F}^k y$ lie in different elements of the Markov partition $\}$. This is more or less the same as the separation time defined above for the same notation. With any $0 < \beta < 1$ the function β^s is a metric providing the original Cantor topology.

On $\bar{\Delta}$ Young uses two kind of norms: the $\mathcal{C}$ norm is

$$\|\varphi\|_{\mathcal{C}} \stackrel{\text{def}}{=} \sup_{l,j} \left\| \varphi|_{\bar{\Delta}_{l,j}} \right\|_\infty e^{-l\epsilon}$$

where $\|\,.\,\|_\infty$ is the essential supremum wrt $\bar{\mu}_\Delta$, and the indices (l,j) refer to the elements of the Markov partition. The $\mathcal{L}$ norm is a sum of this, and the h-norm:

$$\|\varphi\|_h \stackrel{\text{def}}{=} \sup_{l,j} \left(\sup_{x,y \in \bar{\Delta}_{l,j}} \frac{|\varphi(x) - \varphi(y)|}{\beta^{s(x,y)}} \right) e^{-l\epsilon};$$

where the inner sup is again essential supremum wrt $\bar{\mu}_\Delta \times \bar{\mu}_\Delta$. To a Hölder function on the original billiard phase-space, we can associate a function on $\bar{\Delta}$ as described above, such that for any β smaller than a certain number (computed from the original Hölder exponent) the resulting function has a finite h-norm.

In these definitions the role of ϵ is the following: without ϵ the Jacobian of the mapping would be 1 except when recurring to the base of the tower. However estimates expressed in the terms of this norm see a uniform expansion. To make the mapping expanding, when recurring to the base, we have to choose ϵ smaller then the Lyapunov exponent.

The Perron-Frobenius (or transfer) operator P is defined on functions on $\bar{\Delta}$ with finite $\mathcal{L}$-norm as the adjoint of $\bar{F}$ wrt the measure $\bar{\mu}_\Delta$. This means

$$P(\varphi)(x) = \sum_{y | \bar{F}y = x} \frac{\varphi(y)}{J(y)}$$

where J is the Jacobian i. e. the Radon-Nikodym derivative $\frac{d\bar{F}_*^{-1}\bar{\mu}_\Delta}{d\bar{\mu}_\Delta}$.

Another important operator which is heavily used in referenced theorems is the Fourier transform of the transfer operator:

$$P_t\varphi = P(e^{it\kappa}\varphi)$$

where κ is thee free flight vector, more precisely a function on $\bar{\Delta}$ which is obtained from the free flight function, as desribed above. All these operators are quasicompact on the $\mathcal{L}$-space.

3.4. **Local Limit Theorem and Related results.** Recall that by CLT for the Lorentz process ([BS 81, BCS 91]) there is a positive definite matrix D such that $D^{-1}S_n/\sqrt{n}$ converges to a 2 dimensional standard Gaussian distribution. In fact by using the shorthand $\kappa_n = \kappa(f^n(x))$, we have then

$$D^2 = \mu_0(\kappa_0 \otimes \kappa_0) + 2\sum_{j=1}^{\infty} \mu_0(\kappa_0 \otimes \kappa_n). \tag{5}$$

The importance of the Fourier transform operator, is that

$$\int e^{itS_n} d\bar{\mu}_\Delta = \int P^n(e^{itS_n} d\bar{\mu}_\Delta = \int P_t^n(\mathbf{1}) d\bar{\mu}_\Delta.$$

Spectral analysis of the Fourier transform operator leads to the understanding of the characteristic function of the sum S_n.

The results of the spectral analysis can be summarized in the following theorem proved in [SzV 04].

Proposition 3.2. *There are constants $\epsilon > 0$, $K > 0$, and $\theta < 1$ such that*

(a) There are functions $\rho_t : [-\epsilon, \epsilon]^2 \to \mathcal{L}$ and $\lambda_t : [-\epsilon, \epsilon]^2 \to \mathbb{C}$, such that

$$\left\| P_t^n(h) - \lambda_t^n \rho_t \int h d\bar{\mu}_\Delta \right\|_{\mathcal{L}} \le K\theta^n \|h\|_{\mathcal{L}}$$

for all $h \in \mathcal{L}$, $|t| < \epsilon$, $n > 0$. Moreover $\rho_t = 1 + O(t)$, and $\lambda_t = 1 - D\frac{t^2}{2} + o(t^2)$ as $t \to 0$.

(b) For $t \notin [-\epsilon, \epsilon]^2$ we have

$$\|P_t^n(h)\|_{\mathcal{L}} \le K\theta^n \|h\|_{\mathcal{L}}$$

for all $n > 0$.

These tools has been used in [SzV 04] to obtain the following result.

Proposition 3.3. *Let x be distributed on Ω_0 according to μ_0. Let the distribution of $m(S_n(x))$ be denoted by Υ_n. There is a constant $\mathbf{c}$ such that*

$$\lim_{n\to\infty} n\Upsilon_n \to \mathbf{c}l$$

where l is the counting measure on the integer lattice $\mathbb{Z}^2$ and $\to$ stands for vague convergence.

Remark. *In fact,* $\mathbf{c} = \frac{1}{2\pi\sqrt{\det D^2}}$.

The following result is a slight extension of Theorem 4.2 of [SzV 04] and can be proven similarly.

Proposition 3.4. *For each fixed k the following holds.*
If $n_1, n_2 \dots n_k \to \infty$, then

$$\mu_0(m(S_{n_1}) = m(S_{n_1+n_2}) = \dots = m(S_{n_1+n_2+\dots+n_k}) = 0) \sim \prod_{j=1}^{k} \frac{\mathbf{c}}{n_j}.$$

3.5. **Properties of standard pairs.** In the sequel we are still considering billiards (Ω_0, f_0, μ_0) and functions $A : \Omega_0 \to \mathbb{R}^d$, most frequently with $d = 2$. Let us introduce the space of functions (over (Ω_0, f_0, μ_0)) we are to consider. Take $\theta < 1$ close to 1. Let $s(x, y)$ be the smallest n such that either $f_0^n x$ and $f_0^n y$ or $f_0^{-n} x$ and $f_0^{-n} y$ are separated by a singularity. Define the dynamical Hölder space of functions $A : \Omega_0 \to \mathbb{R}$

$$\mathcal{H} = \{A : |A(x) - A(y)| < \text{Const}\theta^{s(x,y)}\}.$$

Let $A_n(x) = \sum_{j=0}^{n-1} A(f_0^j x)$.

Proposition 3.5. *Let ℓ be a standard pair, $A \in \mathcal{H}$ and take n such that $|\log \text{length}(\ell)| < n^{1/2-\delta}$. Then the following statements hold true:*
(a) There is a constant such that

$$\left| \mathbb{E}_\ell(A \circ f_0^n) - \int A d\mu_0 \right| \le \text{Const}\theta^n |\log \text{length}(\ell)|$$

(b) Let $A, B \in \mathcal{H}$ have zero mean. Then

$$\mathbb{E}_\ell(A_n B_n) = n\sigma_{A,B} + O(|\log^2 \text{length}(\ell)|)$$

where

$$\sigma_{A,B} = \sum_{j=-\infty}^{\infty} \int A(x) B(f_0^j x) d\mu_0(x).$$

(c) Let x be distributed according to ℓ and $w_n(t)$ be defined by

$$w_n\left(\frac{i}{n}\right) = \frac{S_i}{\sqrt{n}}$$

with linear interpolation in between. (S_i is the notation for partial sums of the mean free path from the Introduction). Then, as $n \to \infty$, w_n converges weakly (in $C([0, 1] \to \mathbb{R}^2)$) to the 2 dimensional Brownian Motion with zero mean and covariance matrix D^2 given by (5). A similar result holds for LLP.

(d) If $1 < R < n^{1/6-\delta}$ *then*

$$\mathbb{P}_\ell(|A_n - n\int A d\mu_0| \geq R\sqrt{n}) \leq c_1 e^{-c_2 R^2}.$$

(e) If A *is a continuous function on* Ω *supported on* Ω_0 *then*

$$n\mathbb{E}_\ell(A \circ f^n) \to \mathbf{c} \int A d\mu_{(0)}$$

where $\mathbf{c}$ *is the constant from subsection 3.3. For LLP we have*

$$\sqrt{n}\mathbb{E}_\ell(A \circ f^n) \to \mathbf{c}^* \int A d\mu_{(0)}$$

Parts (a) and (c) are proven in [Ch 06]. For part (b) see Lemma 5.12 of [CD 05]. (The error estimate of part (b) is not stated explicitly in [CD 05] but it can be easily deduced from the proof of Lemma 5.12.) Part (d) is proven in [CD 05], Section A.4 for a particular A but the proof in the general case is exactly the same. Part (e) follows from Proposition 3.3 by approximating δ-functions on unstable curves by Hölder functions.

3.6. **Coupling.** A coupling approach introduced in [You 99] is powerful tool for studying Sinai billiards. Here we extend this method to Lorentz process. We begin with a preliminary result. Assume $A \in \mathcal{H}$.

Lemma 3.6. *Given* $\delta_0 > 0$ *there exist* $q > 0$ *and* $n \geq 1$ *such that for arbitrary pair of standard pairs* $\ell_1 = (\gamma_1, \rho_1), \ell_2 = (\gamma_2, \rho_2)$ *satisfying*

$$\mathbb{P}_{\ell_1}(m(x) = m) = \mathbb{P}_{\ell_2}(m(x) = m) = 1 \tag{6}$$

and $\text{length}(\ell_j) \geq \delta_0$*, there exist probability measures* ν_1 *and* ν_2 *and a constant* c*, and there exist families of standard pairs* $\{\ell_{\beta j}\}_\beta$ *and of constants* $\{c_{\beta j}\}_\beta :\ j = 1, 2$*, satisfying*
(i)

$$\mathbb{E}_{\ell_j}(A \circ f^n) = c\nu_j(A) + \sum_{\beta j} c_{\beta j} \mathbb{E}_{\ell_{\beta j}}(A) \qquad j = 1, 2$$

with $c \geq q$;
(ii) There exists a measure preserving map $\pi : (\gamma_1, \nu_1) \to (\gamma_2, \nu_2)$ *such that*

$$d(f^n x, f^n \pi x) \leq C\theta^n \tag{7}$$

(iii) and finally for every $\rho > 0$

$$\sum_{\beta:\ \text{length}(\ell_{\beta j}) < \rho} c_{\beta j} \leq \text{Const}(\delta_0)\rho \qquad j = 1, 2$$

We shall say that subsets of mass c of ℓ_1 and ℓ_2 are coupled to each other.

Observe that part (ii) of Lemma 3.6 implies in particular that

$$|\nu_1(A \circ f^\tau) - \nu_2(A \circ f^\tau)| \leq K||A||_{\mathcal{H}}\theta^{|m|}.$$

Lemma 3.6 is proven in [Ch 06] for f_0 in place of f but the proof shows that the same result holds for f since if $f^k(x)$ and $f^k(\pi x)$ are close and projections of their orbits to Sinai billiard stay close for $n \geq k$ then the orbits themselves are close.

We observe that for Sinai billiards Lemma 3.6 can be used recursevely to establish exponential mixing. For Lorentz gas we can not do it since we are unable to propogate condition (6). Instead we use the local limit theorem to ensure (6) which implies that correlations go to 0 albeit at a slower rate.

3.7. **Random walks.** A *simple symmetric random walk (SSRW)* on $\mathbb{Z}^1$ is the sequence of partial sums

$$\mathcal{S}_n = \mathcal{X}_1 + \cdots + \mathcal{X}_n$$

where $\mathcal{X}_j$ are i.i.d. taking values ± 1 with probability 1/2 each.

We shall use the following well-known elementary properties of SSRW.

Proposition 3.7. *(a) For any $A, B > 0$, P(SSRW visits A before $-B$) $= \frac{B}{A+B}$.*

(b)

$$\mathbb{P}(\mathcal{S}_1 \geq 0, \ldots, \mathcal{S}_{2n} \geq 0) = \binom{2n}{n}\frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}.$$

(c) As $n \to \infty$, $\frac{\mathcal{S}_n}{\sqrt{n}}$ converges to a Gaussian random variable with zero mean and variance 1/4.

(d) There is a constant $\theta < 1$ such that

$$\mathbb{P}\left(-B < \mathcal{S}_n < A \text{ for } n = 1 \ldots N\right) \leq \theta^{[N/\max(A,B)]}.$$

Parts (a)–(c) are standard (see e.g. [Fel 57], Section III.4). To prove (d), let $k = [N/\max(A, B)]$. For $k = 1$ the result follows from (c) and for general k it follows by induction using the Markov property of the SSRW.

4. Proof of Theorem 1

Proof. We are going to define partial transfer operators on the factorized Young tower of the Sinai billiard $\bar{\Delta}$. Let

$$U_k(\phi) = P^k(\phi \mathbf{1}_{S_k=0}), \quad F_j(\phi) = P^j(\phi \mathbf{1}_{\tau=j}), \quad R_k = \sum_{j>k} F_j.$$

We define a stopping time

$$\nu_n = \min\{l > n \mid S_l = 0\}.$$

Then the following identity holds:

$$\left(\bar{F}^{\nu_n}\right)^* = \sum_{k=0}^{n} R_{n-k} U_k. \tag{8}$$

We need to estimate $\mathbb{P}(\tau > n) = \int R_n(\mathbf{1})$. The proof will follow classical renewal theory (cf chapter 16 of [Spi 64]).

Lemma 4.1. *If φ is a Hölder function (with respect to the Young metric) on the factorised Young tower, then*

$$U_n(\varphi) = \frac{\mathbf{c}}{n}\left(\int \varphi d\mu\right)\mathbf{1} + o\left(\frac{1}{n}\right). \tag{9}$$

The error term is meant in the $\mathcal{L}$-norm.

Proof. Since $4\pi^2 \mathbf{1}_{S_k=0} = \iint_{[-\pi,\pi]^2} e^{itS_k} dt$ we have

$$U(\phi) = \frac{1}{4\pi} \iint_{[-\pi,\pi]^2} P_t^n(\phi) dt.$$

Therefore the result follows from Proposition 3.2. □

Using the positivity of transfer operators we get $R_l > R_m$, if $l < m$ in the sense that $R_l - R_m$ is a positive operator. In particular $1 \geq \sum_{k=0}^{n} R_{n-k} U_k(\mathbf{1})$. Using the monotonicity of R_k this sum can be estimated $\sum_{k=0}^{n} R_{n-k} U_k(\mathbf{1}) \geq R_n \sum_{k=0}^{n} U_k(\mathbf{1})$, and so (9) implies:

$$\limsup \mathbf{c} \log n \int R_n(\mathbf{1}) \leq 1.$$

On the other hand $1 \leq \sum_{j=0}^{k} R_{n-k} U_j(\mathbf{1}) + \sum_{j=k+1}^{n} R_0 U_j(\mathbf{1})$. Let us choose $k = k(n) = n - \left[\frac{n}{\log n}\right]$. By (9) the second term in the inequality is $o(1)$, and since $\log k \sim \log n \sim \log(n-k)$ we get

$$\liminf \mathbf{c} \log n \int R_n(\mathbf{1}) \geq 1.$$

The result follows. □

5. Proof of Theorem 2

Proof. We shall show that, for each k, $\mu_0\left(\left(\frac{cN_n}{\log n}\right)^k\right) \to k!$ The proof is by induction on k. For $k = 1, 2$ this is shown in [SzV 04], subection 5.3. We have

$$\mu_0(N_n^k) = \sum_{j_1, j_2 \dots j_k = 1}^{n} \mu_0(m(S_{j_1}) = m(S_{j_2}) = \dots = m(S_{j_k}) = 0). \tag{10}$$

We shall use an elementary estimate which can be proven by induction on k

$$\sum_{i_1+i_2+i_k \le n, i_j \ge 1} \frac{1}{i_1}\frac{1}{i_2}\dots\frac{1}{i_k} \sim (\log n)^k \tag{11}$$

Fix $L \gg 1$. Then by induction the contribution to (10) of where there are two indexes at most L apart is bounded by $\mathrm{Const}(L)(\log n)^{k-1}$. On the other hand Proposition 3.4 together with (11) imply that the contribution of terms where any two indexes are at least L apart

$$k!(\log n)^k(1 + o_{L\to\infty}(1)).$$

This completes the proof. □

Remark. *Similar results for negatively curved surfaces are obtained in* [AD 97]*. Our proof uses moment method of* [DK 57]*. Other approaches (cf* [Aa 81]*,* [Aa 97]*) require more information about the statistical properties of the first return map to* Ω_0.

6. Proof of Theorem 4

Let us describe briefly the idea of the proof. Decomposition (3) shows that it suffices to assume that x is distributed according to some standard pair ℓ satisfying

$$\mathbb{P}_\ell(m(x) = m) = 1, \quad \mathrm{length}(\ell) \ge \frac{1}{|m|^{100}} \tag{12}$$

Proposition 3.5(c) tells us that after the appropriate rescaling $D^{-1}S_n$ converges to a standard 2 dimensional Brownian Motion. Now for the Brownian Motion $w(t)$ it is easy to compute the distribution of time it takes to reach a ball of radius 1 starting from distance R from the origin. Namely, $\log|w(t)|$ is martingale, so $\mathbb{P}(w(t)$ escapes from the ball of radius R^r before reaching the unit ball$) = \frac{\log R}{r \log R} = \frac{1}{r}$.

Since $\sup_{t \le T}|w(t)|$ grows like $\sqrt{T}$ it is easy to see that the limiting distribution of the logarithm of the hitting time rescaled by $\log R$ converges to (2).

Unfortunately S_n can be approximated by a Brownian Motion on the time interval $[0, T]$ with an error which grows like a power of T, only, so this approximation can not directly justify Theorem 4.

To overcome this problem we consider a family of ellipses with geometrically decreasing sizes and use the convergence to the Brownian Motion to estimate the passage of each individual annulus. The invariance of the standard pair given by Proposition 3.1(c)-(d) plays a key role in our analysis.

Now we give the formal proof.

Proof of Theorem 4. Denote $||m|| = |D^{-1}m|$ where $|\cdot|$ is the standard Euclidean distance.

We need an auxiliary result. Let ℓ be a standard pair satisfying (12). Denote

$$\mathcal{C}_k = \{x : |x| = 2^k||m||\}, \quad k \in \mathbb{Z}$$

and

$$\mathcal{D}_k = \{x : |x| \le 2^k||m||\}, \quad k \in \mathbb{Z}.$$

Let H be the maximal free flight (in the $||\cdot||$ norm). Define $s_j(x)$ as follows. $s_0(x) = 0$ and if $s_j(x)$ is already defined so that $d(S_{s_j}, \mathcal{C}_k) \le H$ for some k, then let $s_{j+1}(x)$ be the first time after $s_j(x)$ such that either $||m(S_n)|| \le 2^{k-1}||m||$ or $||m(S_n)|| \ge 2^{k+1}||m||$.

Lemma 6.1. *The following estimates hold uniformly for all standard pairs satisfying (12)*

$$(a) \quad \mathbb{P}_\ell(s_1 > n) \le \text{Const}\min\left(\theta^{n/m^2} + \frac{1}{|m|^{100}}, \theta^{n/(m^2\ln n)} + \frac{1}{n^{100}}\right).$$

(b) For all $\delta > 0$ $\mathbb{P}_\ell(s_1 < m^{2-\delta}) < \frac{1}{|m|^{100}}$.

(c) For a suitable $\zeta > 0$ one has

$$\mathbb{P}_\ell(\ ||m(x_{s_1})|| \le ||m(x)||/2\ ) = \frac{1}{2} + O(|m|^{-\zeta}).$$

(d)

$$\mathbb{E}_\ell(A \circ f^{s_1}) = \sum_\alpha c_\alpha \mathbb{E}_{\ell_\alpha}(A)$$

where ℓ_α are standard pairs and

$$\sum_{\text{length}(\ell_\alpha) < |m|^{-100}} c_\alpha = O(|m|^{-97}).$$

The proof of Lemma 6.1 is given in the next section. Here we deduce the theorem from the lemma.

Lemma 6.1 allows us to approximate $\log_2 ||S_{s_n}||$ by a random walk. This estimate works well if $||S_{s_n}||$ is large. Next we prove an *a priori* estimate which will be used to handle the case when $||S_{s_n}||$ is small.

Lemma 6.2. *Let* $\mathbf{n}$ *be the largest number such that* $s_{\mathbf{n}} < \tau(x)$*. Then there exists* $C > 0$ *such that* $\mathbb{P}_\ell(\mathbf{n} > C \log^5 |m|) \to 0$ *as* $m \to \infty$*.*

Proof. Let $h)0 = 0$, and the ladder index h_j be the first time ($h \in \mathbb{Z}_+$) when $||m(S_{s_h})|| < ||m(x)||/2^j$. We claim that

$$\mathbb{P}_\ell(h_1 > \log^4 |m|) \leq \frac{\text{Const}}{\log^2 |m|}. \tag{13}$$

We note that here we can also use $\log_2 ||m||$.

Indeed parts (c) and (d) of Lemma 6.1 imply that $\log ||S_{s_n}||$ can be well approximated by a random walk in the sense that for any sequence of ups and downs of length n the probability that $\log ||S_{s_j}||$, $j \leq n$ follows this sequence is

$$\frac{1}{2^n}\left(1 + O\left(n 2^{k\zeta} |m|^{-\zeta}\right)\right)$$

where $-k$ is the minimum of the corresponding walk. In our case $k = 1$. Now (13) follows from Proposition 3.7(b).

Combining (13) with Lemma 6.1(d) we obtain that for any R and any j satisfying $\log_2 ||m|| - 0 \geq j \geq \log_2 ||m|| - \log_2 R$

$$\mathbb{P}_\ell(h_j - h_{j-1} < [\log_2 ||m|| - (j-1)]^4) \geq 1 - \frac{\text{Const}}{[\log_2 ||m|| - (j-1)]^2}. \tag{14}$$

Therefore

$$\mathbb{P}_\ell(\min\{S_k \mid 1 \leq k \leq \sum_{r=\log_2 R}^{\log_2 ||m||} r^4\} < R) \geq 1 - \sum_{j=\log_2 R}^{\log_2 ||m||} \frac{C}{r^2} \geq 1 - \sum_{r=\log_2 R}^{\infty} \frac{C}{r^2}.$$

The last sum can be made as small as we wish by choosing R large.

Moreover by Lemma 6.1(d) given R, ε_1 we can find δ_0 such that if η is the first time when $||S_\eta|| \leq R$ then

$$\mathbb{E}_\ell\left(A \circ f^\eta\right) = \sum_\alpha c_\alpha \mathbb{E}_{\ell_\alpha}(A)$$

where

$$\sum_{\text{length}(\ell_\alpha) \leq \delta_0} c_\alpha \leq \varepsilon_1.$$

Now observe that the set of standard pairs satisfying

$$\mathbb{P}_\ell(m(x) \leq R) = 1, \quad \text{length}(\ell) \geq \delta_0 \tag{15}$$

endowed with topology of weak convergence of $\mathbb{E}_\ell$-measures is compact. Therefore given R, δ_0, ε we can find M such that for any standard pair satisfying (15) we have

$$\mathbb{P}_\ell(\tau > M) < \varepsilon$$

The lemma follows. □

Let $\bar{m} = 2^{\log_2^{1/20} ||m||}$. Using again the approximation by simple random walk we see that the probability that $||m(x_n)||$ reaches $\bar{m}$ before reaching $||m||^r$ is $\frac{r-1}{r} + o(1)$. (Observe that approximation error is $O(n\bar{m}^{-\zeta})$ and by Proposition 3.7 it is enough to restrict our attention to $n \ll \log^3 ||m||$.) On the other hand, by Lemma 6.2, the probability that the particle starting from a ball of radius $\bar{m}$ reaches the ball of radius m^r before time $\tau(x)$ converges to 0 as $|m| \to \infty$. It follows that

$$\mathbb{P}_\ell\left(\max_{k\leq\tau} \log ||m(S_k)|| > r \log ||m||\right) \to \frac{1}{r} \tag{16}$$

By Lemma 6.2, with probability close to 1 we have

$$\max_{j<\mathbf{n}}(s_{j+1} - s_j) \leq \tau(x) \leq C \log^5 ||m|| \max_{j<\mathbf{n}}(s_{j+1} - s_j) \tag{17}$$

By Lemma 6.1(a)&(b), for every fixed $\delta > 0$ the probability that

$$(\max_{j<\mathbf{n}} ||m(S_{s_j}||)^{2-\delta} \leq \max_{j<\mathbf{n}}(s_{j+1} - s_j) \leq (\max_{j<\mathbf{n}} ||m(S_{s_j}||)^{2+\delta} \tag{18}$$

converges to 1 as $|m| \to \infty$.

Combining (16) with (17) and (18) we obtain (2). □

Remark. *The distributions similar to those described in this section appear in the study of random walk on the group of affine transformations of the real line (cf.* [Gr 74]*).*

7. Escape from an annulus.

Proof of Lemma 6.1. For a standard pair $\ell = (\gamma, \rho)$ denote by $[\ell]$ the value of $m \in \mathbb{Z}^2$ for which $\gamma \in \Omega_m$.

The idea of the proof of (a) is borrowed from the inductive proof of Proposition 3.7(d), and is based on Proposition 3.1(d).

We have to prove two inequalities

$$\mathbb{P}_\ell(s_1 > n) \leq \text{Const}\left(\theta^{n/m^2} + \frac{1}{|m|^{100}}\right) \text{ and} \tag{19}$$

$$\mathbb{P}_\ell(s_1 > n) \leq \text{Const}\left(\theta^{n/(m^2 \ln n)} + \frac{1}{n^{100}}\right). \tag{20}$$

To prove (19) it is enough to restrict our attention to $n \leq |m|^3$ since the RHS of (19) stays constant for $n > |m|^3$. Let

$$p_k = \mathbb{P}_\ell(s_1 > k|m|^2).$$

Using the Markov decomposition

$$\mathbb{E}_\ell(A \circ f^{k|m|^2}) = \sum_\alpha c_\alpha \mathbb{E}_{\ell_\alpha}(A),$$

Proposition 3.1(d) and the fact that by Proposition 3.5(c) there is $\theta < 1$ such that for any ℓ with length$(\ell) \geq |m|^{-101}$

$$\mathbb{P}_\ell(s_1 \leq |m|^2) \geq 1 - \theta$$

we obtain

$$p_{k+1} \leq \theta p_k + \mathrm{Const}|m|^{-101}.$$

(19) follows.

The proof of (20) is similar. We replace $k|m|^2$ by $kC|m|^2 \ln n$ for sufficiently large C and use the fact that for any ℓ with length$(\ell) \geq n^{-101}$

$$\mathbb{P}_\ell(s_1 \leq C \ln n |m|^2) \geq 1 - \theta$$

(b) follows from Proposition 3.5(d).

(c) Let $k = ||m||^\zeta$ $0 < \zeta < 1/5$. We consider S_{jk} stopped when either $||m(S_{jk})|| \leq ||m||/2 - Hk$ or $||m(S_{jk})|| \geq 2||m|| + Hk$. Let $\bar{s}$ be the corresponding stopping time. Call $Z = S_{(j+1)k} - S_{jk}$. Let $X_j = \log ||S_{jk}||^2$. Note that

$$||S_{(j+1)k}||^2 = (D^{-1}S_{jk}, D^{-1}S_{jk}) + +2(D^{-1}S_{jk}, D^{-1}Z) + (D^{-1}Z, D^{-1}Z).$$

Taking Taylor expansion of

$$X_{j+1} - X_j = \log\left(1 + \frac{2(D^{-1}S_{jk}, D^{-1}Z) + (D^{-1}Z, D^{-1}Z)}{(D^{-1}S_{jk}, D^{-1}S_{jk})}\right)$$

and using the bounds

$$S_{jk} = O(|m|), \quad Z = O(k), \quad |S_{jk}| \geq \frac{m}{2} - Hk$$

we get

$$X_{j+1} - X_j = \tag{21}$$

$$\frac{2(D^{-1}S_{jk}, D^{-1}Z)}{||S_{jk}||^2} + \frac{(D^{-1}Z, D^{-1}Z)}{||S_{jk}||^2} - 2\frac{(D^{-1}S_{jk}, D^{-1}Z)^2}{||S_{jk}||^4} + O\left(\frac{k^3}{|m|^3}\right).$$

To estimate the expectation of the first term in (21) let $\bar{Z} = S_{jk} - S_{(j-1)k}$. Then

$$\frac{(D^{-1}S_{jk}, D^{-1}Z)}{||S_{jk}||^2}$$

$$= \frac{(D^{-1}S_{(j-1)k}, D^{-1}Z)}{||S_{(j-1)k}||^2} + \frac{(D^{-1}\bar{Z}, D^{-1}Z)}{||S_{(j-1)k}||^2} -$$

$$-2\frac{(D^{-1}S_{(j-1)k}, D^{-1}Z)(D^{-1}S_{(j-1)k}, D^{-1}\bar{Z})}{||S_{(j-1)k}||^4} + O\left(\frac{k^3}{|m|^3}\right)$$

$$= I + II + III + O\left(\frac{k^3}{|m|^3}\right).$$

By Proposition 3.1(c)

$$\mathbb{E}((A \circ f_0^{(j-1)k})\mathbf{1}_{\bar{s}>(j-1)k}) = \sum_\alpha c_\alpha \mathbb{E}_{\ell_\alpha}(A).$$

Take some α. Observe that on γ_α, $S_{(j-1)k}$ equals to a constant, say S_α, where

$$\frac{||m||}{2} - Hk \le ||S_\alpha|| \le 2||m|| + Hk.$$

By Proposition 3.5(a)

$$\mathbb{E}_{\ell_\alpha}(I) = O\left(\frac{1}{|m|}\theta^k \log \mathrm{length}(\ell_\alpha)\right)$$

$$\mathbb{E}_{\ell_\alpha}(II) = O\left(\frac{1}{|m|^2} \log \mathrm{length}(\ell_\alpha)\right)$$

$$\mathbb{E}_{\ell_\alpha}(III) = O\left(\frac{||S_\alpha||^2}{|m|^4} \log \mathrm{length}(\ell_\alpha)\right) = O\left(\frac{1}{|m|^2} \log \mathrm{length}(\ell_\alpha)\right).$$

Using Proposition 3.1(d) and Abel resummation formula we get

$$\mathbb{E}_\ell\left(\frac{2(D^{-1}S_{jk}, D^{-1}Z)}{||S_{jk}||^2}\mathbf{1}_{\bar{s}>(j-1)k}\right) = O\left(\frac{1}{m^2}\right). \tag{22}$$

To estimate other terms in (21) we use the decomposition

$$\mathbb{E}((A \circ f_0^{jk})\mathbf{1}_{\bar{s}>jk}) = \sum_\beta c_\beta \mathbb{E}_{\ell_\beta}(A).$$

Proposition 3.5(b) gives

$$\mathbb{E}_{\ell_\beta}\left((D^{-1}Z, D^{-1}Z)\right) = \sum_{p,q=1}^{2} D^{-2}_{pq}\mathbb{E}_{\ell_\beta}(Z_{(p)}Z_{(q)})$$

$$= k\sum_{p,q=1}^{2} D^{-2}_{pq}D^2_{pq} + O(\log^2 \mathrm{length}(\ell_\beta)) = 2k + O(\log^2 \mathrm{length}(\ell_\beta))$$

and

$$\mathbb{E}_{\ell_\beta}\left((D^{-1}S_\beta, D^{-1}Z)^2\right) = \sum_{p_1,q_1,p_2,q_2=1}^{2} D^{-2}_{p_1q_1}D^{-2}_{p_2q_2}S_{\beta(p_1)}S_{\beta(p_2)}\mathbb{E}_{\ell_\beta}(Z_{(q_1)}Z_{(q_2)}) =$$

$$= k \sum_{p_1,q_1,p_2,q_2=1}^{2} D^{-2}_{p_1q_1} D^{-2}_{p_2q_2} S_{\beta(p_1)} S_{\beta(p_2)} D^2_{q_1q_2} + O(\log^2 \text{length}(\ell_\beta)) =$$

$$= ||S^2_\beta||k + O(\log^2 \text{length}(\ell_\beta)).$$

Hence

$$\mathbb{E}_{\ell_\beta}\left(\left[\frac{(D^{-1}Z, D^{-1}Z)}{||S_{jk}||^2} - 2\frac{(D^{-1}S_{jk}, D^{-1}Z)^2}{||S_{jk}||^4}\right]\mathbf{1}_{\bar{s}>jk}\right) = O\left(\frac{\log^2 \text{length}(\ell_\beta)}{|m|^2}\right).$$

Using Proposition 3.1(d) and Abel resummation formula we get

$$\mathbb{E}_{\ell}\left(\left[\frac{(D^{-1}Z, D^{-1}Z)}{||S_{jk}||^2} - 2\frac{(D^{-1}S_{jk}, D^{-1}Z)^2}{||S_{jk}||^4}\right]\mathbf{1}_{\bar{s}>jk}\right) = O\left(\frac{1}{|m|^2}\right).$$

Combining this with (22) we get

$$\mathbb{E}_\ell((X_{j+1} - X_j)\mathbf{1}_{\bar{s}>jk}) = O\left(\frac{1}{|m|^2}\right).$$

It follows that

$$\mathbb{E}_\ell(X_{\min(\bar{s},|m|^{2+\delta})}) = 2\log||m|| + O\left(\frac{1}{||m||^{\zeta-\delta}}\right).$$

On the other hand by part (a)

$$X_{\min(\bar{s},|m|^{2+\delta})} = 2\log||m|| \pm \log 4 + O\left(\frac{\log||m||}{||m||^{1-\zeta}}\right)$$

except on the set of probability $O(|m|^{-100})$. Therefore

$$\mathbb{P}_\ell(||m(S_{\bar{s}})|| \leq ||m||/2 - Hk) = \frac{1}{2} - O\left(\frac{1}{|m|^{\zeta-\delta}}\right).$$

Since $||m(S_{s_1})|| \leq ||m||/2$ if $||m(S_{\bar{s}})|| \leq ||m||/2 - Hk$, we conclude that

$$\mathbb{P}_\ell(||m(S_{s_1})|| \leq ||m||/2) \geq \frac{1}{2} - O\left(\frac{1}{|m|^{\zeta-\delta}}\right).$$

A similar argument shows that

$$\mathbb{P}_\ell(||m(S_{s_1})|| \geq 2||m||) \geq \frac{1}{2} - O\left(\frac{1}{|m|^{\zeta-\delta}}\right).$$

This proves (c).

To prove (d) observe that each $\gamma_\alpha = \gamma_j(x)$ and by part (a)

$$\mathbb{P}_\ell(j > |m|^3) \leq \frac{1}{|m|^{100}}$$

Now the result follows by Proposition 3.1 (d).

□

8. Proof of Theorem 5

Proof. In view of decomposition (3) it suffices to show that if $|m_1|, |m_2| \to \infty$ and if ℓ_1, ℓ_2 are standard pairs such that

$$P_{\ell_j}(m(x) = m_j) = 1, \quad \text{length}(\ell_j) > |m_j|^{-100} \tag{23}$$

then

$$|\mathbb{E}_{\ell_1}(A(f^\tau(x))) - \mathbb{E}_{\ell_2}(A(f^\tau(x)))| \to 0 \tag{24}$$

as $|m_1|, |m_2| \to \infty$.

We claim that it can be assumed without the loss of generality that

$$\frac{1}{2} < \frac{|m_1|}{|m_2|} < 2. \tag{25}$$

Indeed suppose that, say $|m_2| > 2|m_1|$. Let $\tilde\tau(x)$ be the first time such that $|(S_{\tilde\tau(x)})| \leq 2|m_1|$. It suffices to show that

$$\mathbb{E}_{\ell_2}(A \circ f^{\tilde\tau}) = \sum_\alpha c_\alpha \mathbb{E}_{\ell_\alpha}(A)$$

where

$$\sup_{m_1,\ell_2} \sum_{\text{length}(\ell_\alpha)<|m|^{-100}} c_\alpha \to 0 \text{ as } |m_1| \to \infty$$

but this follows easily from the analysis of Section 6 (in particular Lemma 6.1(d)).

To prove (24) we extend the coupling lemma Lemma 3.6 to the Lorentz process.

Lemma 8.1. *Given $\zeta > 0$ and $\varepsilon > 0$ there exists R such that for any two standard pairs $\ell_1 = (\gamma_1, \rho_1), \ell_2 = (\gamma_2, \rho_2)$ satisfying* (23), (25) *and $|m_j| > R$ the following holds.*

Let $\bar n = |m_1|^{2(1+\zeta)}$. There exist positive constants $\bar c$ and $\bar c_\beta$, probability measures $\bar\nu_1$ and $\bar\nu_2$ and families of standard pairs and $\{\bar\ell_{\beta j}\}_{\beta, j=1,2}$ satisfying

$$\mathbb{E}_{\ell_j}(A \circ f^{\bar n}) = \bar c \bar\nu_j(A) + \sum_\beta \bar c_\beta \mathbb{E}_{\bar\ell_{\beta j}}(A) \qquad j = 1, 2 \tag{26}$$

with $\bar c \geq 1 - \varepsilon$ and there exists a measure preserving map $\bar\pi : (\gamma_1, \bar\nu_1) \to (\gamma_2, \bar\nu_2)$ such that for any $n \geq \bar n$ $d(f^n(x), f^n(\bar\pi x)) \leq C\theta^{n-\bar n}$.

Lemma 8.1 implies that

$$|\mathbb{E}_{\ell_1}(A \circ f^\tau) - \mathbb{E}_{\ell_2}(A \circ f^\tau)|$$

$$\leq \left| \sum_\beta c_\beta \left[\mathbb{E}_{\ell_1}(A \circ f^{\tau - \bar n}) - \mathbb{E}_{\ell_2}(A \circ f^{\tau - \bar n}) \right] \right| + \text{Const}||A||_{\mathcal{H}} \theta^{\bar n}$$

$$+(\mathbb{P}_{\ell_1}(\tau < 2\bar{n}) + \mathbb{P}_{\ell_2}(\tau < 2\bar{n}))||A||_\infty$$
$$\leq 2\varepsilon||A||_\infty + \text{Const}||A||_{\mathcal{H}}\theta^{\bar{n}} + (\mathbb{P}_{\ell_1}(\tau < 2\bar{n}) + \mathbb{P}_{\ell_2}(\tau < 2\bar{n}))||A||_\infty$$

Hence (24) follows from Theorem 4 and Lemma 8.1 by choosing ζ and ε sufficiently small. □

It remains to prove Lemma 8.1.

Proof. The Lemma is obtained by a repeated application of Lemma 3.6. Fix $\varepsilon_1 \ll \varepsilon$. By Growth Lemma (Proposition 3.1) we can find a number δ_0 such that if $n > \text{Const}|\log \text{length}(\ell)|$ then

$$\mathbb{E}_\ell(A \circ f^n) = \sum_\alpha c_\alpha \mathbb{E}_{\ell_\alpha}(A)$$

and

$$\sum_{\text{length}(\ell_\alpha)<\delta_0} c_\alpha < \varepsilon_1. \tag{27}$$

Let q be the constant from Lemma 3.6.

Take R such that the probability that the absolute value of a Gaussian random variable with mean 0 and variance D^2 exceeds R is less than ε_1.

Take k such that $(1-\frac{q}{2})^k < \varepsilon$. Take $\bar{\zeta}$ such that $(1+\bar{\zeta})^k < 1+\zeta$. Set $n_j = |m_1|^{2j(1+\bar{\zeta})}$, $j = 1\dots k$. Combining Proposition 3.5(e), (27) and the definition of R we obtain

$$\mathbb{E}_{\ell_j}(A \circ f^{n_1}) = \sum_m \sum_\alpha c_{\alpha m j} \mathbb{E}_{\ell_{\alpha m j}}(A)$$

where $P_{\ell_{\alpha m j}}(m(x) = m) = 1$ and

$$\sum_{|m_1|<|m|<Rm_1^{1+\bar{\zeta}}} \left| \sum_{\text{length}(\ell_{\alpha m 1})<\delta_0} c_{\alpha m 1} - \sum_{\text{length}(\ell_{\alpha m 2})<\delta_0} c_{\alpha m 2} \right| < 100\varepsilon_1.$$

Applying Lemma 3.6 to couple $\sum_\alpha c_{\alpha m 1}\mathbb{E}_{\ell_{\alpha m 1}}$ and $\sum_\alpha c_{\alpha m 2}\mathbb{E}_{\ell_{\alpha m 2}}$ we obtain

$$\mathbb{E}_{\ell_j}(A\circ f^{n_1}) = \sum_m c_m \nu_{jm}(A) + \sum_\beta c_{\beta j}\mathbb{E}_{\ell_{\beta j}}(A) + \sum_\kappa c_{\kappa j}\mathbb{E}_{\ell_{\kappa j}}(A) \qquad j = 1, 2$$

where ν_{m1} and ν_{m2} satisfy the conditions of Lemma 3.6,

$$\sum_m c_m > q - 100\varepsilon_1, \quad \sum_\kappa c_{\kappa j} < 100\varepsilon_1$$

and

$$\mathbb{P}_{\ell_{\beta j}}(|m(x)| < R|m_1|^{1+\bar{\zeta}}) = 1.$$

Splitting each $c_{\beta j}$ into several pieces if necessary we can assume that $c_{\beta 1} = c_{\beta 2}$. Next we apply the same procedure with ℓ_1, ℓ_2 replaced by $\ell_{\beta 1}, \ell_{\beta 2}$ and n_1 replaced by n_2. Continuing this k times we obtain (26) with

$$1 - \bar{c} \leq (1 - q + 100\varepsilon_1)^k + 100k\varepsilon_1.$$

The result follows. □

9. Continuous time.

Here we prove Corollaries 6 and 7. Let $r : \Omega \to \mathbb{R}_+$ be the free path length. If $A \in L^1(\mathbf{m})$ let $\bar{A} = \int_0^r A(g^s x) ds$. Then

$$\mu(\bar{A}) = \mathbf{m}(A)\bar{L}.$$

Let $p : M \to \Omega$ be the place of the first backward collision.

Given t let $n(t)$ be the number such that $T_n \leq t < T_{n+1}$. By the ergodicity of (Ω_0, f_0, μ_0) we have $T_n/n \to \bar{L}$ almost surely. In other words

$$\frac{n(t)}{t} \to \frac{1}{\bar{L}} \tag{28}$$

almost surely.

Proof of Corollary 6. By Ratio Ergodic Theorem it suffices to prove Corollary 6 for one function A. In particular we can assume that A is positive and bounded. Then

$$\int_0^t A(g^s) ds = \sum_{j=0}^{n(t)-1} \bar{A}(f^j p(x)) + O(1).$$

Hence it is enough to show that

$$\frac{\sum_{j=0}^{n(t)-1} \bar{A}(f^j p(x))}{\ln t}$$

converges to the exponential random variable with mean $\bar{L}\mathbf{m}(A)/\mathbf{c}$. By (28)

$$\mathbb{P}\left(\frac{\sum_{j=0}^{t/(2L)} \bar{A}(f^j p(x))}{\ln t} \leq \frac{\sum_{j=0}^{n(t)-1} \bar{A}(f^j p(x))}{\ln t} \leq \frac{\sum_{j=0}^{2t/L} \bar{A}(f^j p(x))}{\ln t} \right) \to 1$$

as $t \to \infty$. By Corollary 3 both the first and the third term in the last formula converge to the exponential random variable with mean $\bar{L}\mathbf{m}(A)/\mathbf{c}$. The result follows. □

Proof of Corollary 7.

$$\frac{\ln t_m}{\ln |m|} = \frac{\ln n(t_m)}{\ln |m|} + \frac{\ln \left(\frac{t_m}{n(t_m)}\right)}{\ln |m|}.$$

□

10. Linear Lorentz process: limit theorems.

Theorem 9 follows from the local limit theorem for LLP and the relation

$$\sum_{n_1+n_2+\dots+n_k=n} \prod_j \frac{1}{\sqrt{n}_j}$$
$$\sim k! n^{k/2} \int \dots \int_{t_1<t_2<\dots<t_{k-1}<1} \frac{1}{\sqrt{t_1}} \frac{1}{\sqrt{t_2-t_1}} \dots \frac{1}{\sqrt{t_{k-1}-t_{k-2}}} \frac{1}{\sqrt{1-t_{k-1}}} dt_1 \dots dt_{k-1}$$
$$= k! n^{k/2} \frac{\Gamma(1/2)^n}{\Gamma(\frac{n}{2}+1)}$$

which can be proven by induction.

Observe that in fact it is not nesseccary to ask that $x_0 \in \Omega_0$ it suffices to assume that $m(S_0) \ll \sqrt{n}$. Also the similar result holds for $\mathrm{Card}(j \le n : S_j \in O)$.

We now prove theorem 10. Let $\bar{\tau}$ be the first time $m(S_{\bar{\tau}}) = 0$ and let $\bar{\tau}_m$ be the distribution of $\bar{\tau}$. By the remark above

$$\frac{\tau^*_m - \bar{\tau}_m}{|m^2|} \Rightarrow 0$$

so it suffice to prove Theorem 10 with τ_m replaced by $\bar{\tau}_m$. But then the result follows from the functional Central Limit Theorem.

The proof of Theorem 11 is similar to but easier than the proof of Theorem 5. Indeed here we can assume that

$$|d(\ell_1, O) - d(\ell_2, O)| \ll d^\alpha(\ell_1, O)$$

for some $\alpha < 1$ but then $d(\ell_1, \ell_2) \ll d^\alpha(\ell_1, O)$ so we can easily couple ℓ_1 and ℓ_2.

11. Linear Lorentz process: return time tail.

To prove Theorem 8 we need an auxilary fact. Let t_m be the fist time $m(S_t) = m$.

Lemma 11.1. *(a) For any standard pair ℓ there exists the limit*

$$\bar{\mathbf{c}}(l) = \lim_{n\to\infty} n \mathbb{P}_\ell(t_n < \tau^*).$$

(b) There exists a constant C_1 such that for any standard pair ℓ

$$\mathbb{P}_\ell(t_n < \tau^*) \leq \frac{C_1 \ln(\text{length}(\ell))}{n}.$$

(c) There exists a constant C_2 such that for any standard pair ℓ for any number $K \geq 1$

$$\mathbb{P}_\ell(t_n < \tau^* \text{ and } t_n \geq Kn^2) \leq \frac{C_2 \ln(\text{length}(\ell))}{K^{100} n}.$$

To deduce Theorem 8 from the lemma take a small ε. Then

$$\mathbb{P}(\tau > n) \geq \mathbb{P}(t_{\varepsilon\sqrt{n}} < t_0)\mathbb{P}(\tau(x_{t_{\varepsilon\sqrt{n}}}) > n | t_{\varepsilon\sqrt{n}} < t_0).$$

The first factor is asymptotic to $\frac{c_1}{\varepsilon\sqrt{n}}$ by the lemma while the second factor is asymptotic to the probability that the maximum of the standard Brownian motion on the unit interval is less than ε. The last probability is asymptotic to $\frac{c_2}{\varepsilon}$. Thus

$$\liminf \mathbb{P}(\tau > n) \geq c_1 c_2.$$

Take $\bar{\varepsilon} \gg \varepsilon$. To get an estimate from above we need to take into account the probability that for some $p \geq 1$ we have $t_{\varepsilon\sqrt{n}} \in [p\bar{\varepsilon}n, (p+1)\bar{\varepsilon}n]$ in which case it sufficies that

$$\tau(x_{t_{\varepsilon\sqrt{n}}}) > n(1 - \bar{\varepsilon}p).$$

However by Lemma 11.1

$$\mathbb{P}(t_{\varepsilon\sqrt{n}} < t_0 \text{ and } t_{\varepsilon\sqrt{n}} \in [p\bar{\varepsilon}n, (p+1)\bar{\varepsilon}n])$$

is less than Const $\left(\frac{\varepsilon}{p\bar{\varepsilon}}\right)^{100}$ so we can neglect contributions for $p \neq 0$ proving Theorem 8.

Proof of Lemma 11.1. Fix $\delta_0 > 0$. We claim that given ε_0 there exists m_0 such that for all m such that $|m| \geq m_0$ for all ℓ such that

$$\mathbb{P}_\ell(m(x) = m) = 1 \text{ and } \text{length}(\ell) \geq \delta_0$$

we have

$$1 - \varepsilon_0 \leq \frac{|n|}{|m|}\mathbb{P}_\ell(t_n < \tau^*) \leq 1 + \varepsilon_0. \tag{29}$$

To derive parts (a) and (b) of Lemma 11.1 let $\mathbf{t}$ be the first time when

$$m(S_t) \geq m_0 \text{ and } r_t(x) \geq \delta_0$$

and apply (29) to each homogenuos component of $f^{\mathbf{t}}\ell$. (Observe that by Lemma 6.1(a)

$$\mathbb{P}(m(S_j) \leq m_0 \text{ for } j = 1 \dots n) \leq \text{Const}\left[\theta^{n/(m_0^2 \ln n)} + \frac{1}{|n|^{100}}\right]\Bigg).$$

To prove the estimate from below in (29) observe that similarly to Lemma 6.1 we have

$$\mathbb{P}_\ell(t_{2|m|} < \tau^* \text{ and } r_{t_{2|m|}}(x) \geq \frac{1}{|m|^{100}}) = \frac{1}{2} + O(|m|^{-\zeta}).$$

Iterating this estimate we obtain

$$\mathbb{P}_\ell\left(t_{2^k|m|} < \tau^* \text{ and } r_{t_{2|m|}}(x) \geq \frac{1}{(2^k|m|)^{100}}\right) \tag{30}$$

$$= \left(\frac{1}{2}\right)^k \prod_{j=1}^{k} (1 + O((2^k|m|)^{-\zeta})).$$

Taking $k = \log_2(|n|/|m|)$ we obtain the lower bound in (29).

To prove the upper bound let n_1 be the first time then

$$\text{either } S_n \in O \text{ or } r_n(x) \geq \delta_0 \text{ and } m(S_n) \geq 2|m|.$$

If $m(S_{n_j}) \neq 0$ define n_{j+1} be the first time then

$$\text{either } S_n \in O \text{ or } r_n(x) \geq \delta_0 \text{ and } m(S_n) \geq 2m(S_{n_j}).$$

Then by induction

$$\mathbb{P}_\ell(S_{n_k} \in O) = \left(\frac{1}{2}\right)^k [1 + O(|m|^{-\zeta})].$$

On the other hand let $k = \log_2(\sqrt{n}/|m|)$. We have

$$\frac{m(S_{n_k})}{2^k|m|} = \prod_{j=0}^{k-1} \frac{m(S_{n_{j+1}})}{2m(S_{n_j})}.$$

Therefore

$$\mathbb{P}_\ell\left(\ln\left(\frac{m(S_{n_k})}{2^k|m|}\right) \geq r\right) \leq \sum_j \mathbb{P}_\ell\left(\ln\left(\frac{m(S_{n_{j+1}})}{2m(S_{n_j})}\right) \geq \frac{r\theta^j}{1-\theta}\right)$$

$$\leq \sum_j O(\exp(-\text{Const}(\theta^j 2^j r|m_0|))) = O(\exp(-\text{Const} m_0))$$

where the penultimate estimate follows from the Growth Lemma. Next for any ℓ with

$$\text{length}(\ell) \geq \delta_0 \quad \mathbb{P}_\ell(m(x) = \bar{m}) = 1$$

for some $\bar{m} \geq \sqrt{n}$ we have

$$\mathbb{P}_\ell(t_n < \tau^*) - \mathbb{P}_\ell(t_n < \tau^* \text{ and } r_j(x) \geq \frac{1}{n^{100}} \text{ for } j = 1 \dots t_n) = O(n^{-97})$$

so similarly to Section 6 we have

$$\mathbb{P}_\ell(t_n < t_0) = \frac{\bar{m}}{n}(1 + O(|n|^{-\zeta})).$$

Thus similarly to the way we derive the Theorem 8 from Lemma 11.1 we can show that the contribution to (29) of the terms where $r_{t_{2^k|m|}}(x)$ is small for some k can be neglected. (29) follows.

It remains to prove part (c) of Lemma 11.1. To this end observe that $t_n \geq Kn^2$ implies that there is j such that

$$t_{n/2^j} - t_{n/2^{j+1}} \geq \frac{\theta^j K n^2}{1-\theta}.$$

To estimate the probability of such an event we replace one of the 1/2 factors in (30) by

$$\mathbb{P}_\ell\left(t_{n/2^j} - t_{n/2^{j+1}} \geq \frac{\theta^j K n^2}{1-\theta}\right).$$

By Lemma 6.1(a) the last expression is

$$O\left(\theta^{K(2\theta)^j}\right)$$

and part (c) follows. □

Remark. The constant $\bar{\mathbf{c}}$ can be computed using the local limit theorem.

Indeed consider the identity

$$\sum_{j=0}^{n} \mathbb{P}(S_j \in O \text{ and } S_k \notin O \text{ for } k = j+1 \dots n) = 1.$$

Observe that the proof of Lemma 11.1 shows that $\bar{\mathbf{c}}(\ell)$ depends continuously on ℓ.

Lemma 11.2. *There is a constant C such that for all n we have*

$$\mu_0(S_n \in O \text{ and } r_n(x) \leq \delta) \leq \frac{C\delta}{n}.$$

Proof. If $r_n(x) \leq \delta$ then an orbit of x passes close to the singularity near time n. Namely

$$\{r_n(x) \leq \delta\} \subset \bigcup_j \{d(x_{n-j}, \mathcal{S}) \leq \theta^j\}.$$

So we need to estimate

$$\mu_0(S_n \in O \text{ and } d(x_{n-j}, \mathcal{S}) \le \theta^j).$$

By the time reversal symmetry the last expression is the same as

$$\mu_0(S_0 \in O : \quad m(S_n) = 0 \text{ and } d(x_j, \mathcal{S}) \le \theta^j).$$

By the Growth Lemma the contribution of terms with $j > \ln^2 n$ can be neglected. Now by Proposition 3.5(e)

$$\mu_0(S_0 \in O, m(S_n) = 0 \text{ and } d(x_j, \mathcal{S}) \le \theta^j) \le \text{Const}\theta^j/n.$$

□

Now by the local limit theorem given $S_j \in O$ we have that x_j is asymptotically uniformly distributed on $\pi_q^{-1}O$ and by Lemma 11.2 most of the points belong to the long curves. Hence Theorem 8 implies

$$\frac{\bar{\mathbf{c}}}{\sqrt{2\pi}D} \frac{\text{length}(O)}{\text{length}(Q_0)} \sum_j \frac{1}{\sqrt{j}} \frac{1}{\sqrt{n-j}} \sim 1.$$

It follows that

$$\bar{\mathbf{c}} = \frac{\sqrt{2\pi}\sigma}{\Gamma(1/2)^2} \frac{\text{length}(Q_0)}{\text{length}(O)}.$$

12. Two particles.

The proofs of Theorems 12 and 13 are similar to the proofs of Theorems 4 and 5 respectively. The most significant change is that Theorem 4 relies on Lemma 6.1. The proof of part (c) of that lemma, however, uses the exponential mixing for the discrete time system. Since exponential mixing is currently unknown for continuous time system, we indicate a direct proof of Lemma 6.1 for the continuous time system.

We need some notation. Let ℓ_1 and ℓ_2 be standard pairs for x', x'' respectively. We denote $\mathbb{P} = \mathbb{P}_{\ell_1} \times \mathbb{P}_{\ell_2}$. Denote $L = ||m(x'_0) - m(x''_0)||$ and fix a small $\delta > 0$. We say that *some event happens almost certainly* if $\mathbb{P}$–probability of its complement is $O(\theta^{L^\delta})$. Let x'_n, x''_n denote the position of the particles after n collisions.

Let $\hat{n}$ be the first time when either $||m(x'_n) - m(x''_n)|| \ge 2L + L^{0.9}$ or $||m(x'_n) - m(x''_n)|| \le \frac{L}{2} - L^{0.9}$. Then the argument of Lemma 6.1 shows that

$$\mathbb{P}(||m(x'_{\hat{n}}) - m(x''_{\hat{n}})|| \le \frac{L}{2} - L^{0.9}) = 1/2 + O(L^{-\zeta}).$$

Also by Proposition 3.5(d) almost certainly

$$\hat{n} \le L^{2(1+\delta)}. \tag{31}$$

Now let t'_n (t''_n) denote the time it takes the particle x' (x'') to collide n times. Proposition 3.5(d) and (31) imply that almost certainly

$$|t'_{\hat{n}} - t''_{\hat{n}}| \leq L^{(1+\delta)}.$$

It follows that the faster particle almost certainly does not wander farther than $L^{(1+\delta)^3/2}$ from its position at time $\min(t'_{\hat{n}}, t''_{\hat{n}})$ during the time $(\max(t'_{\hat{n}}, t''_{\hat{n}}) - \min(t'_{\hat{n}}, t''_{\hat{n}}))$ it takes the slower particle to collide $\hat{n}$ times. If δ is so small that $L^{(1+\delta)^3/2} < L^{0.9}$, then we have

$$\mathbb{P}(d(x'(t), x''(t)) \text{ reaches } L/2 \text{ before } 2L) \geq 1/2 + O(L^{-\zeta}).$$

Likewise

$$\mathbb{P}(d(x'(t), x''(t)) \text{ reaches } 2L \text{ before } L/2) \geq 1/2 + O(L^{-\zeta}).$$

This establishes Lemma 6.1 for continuous time system. The rest of the proof of Theorem 12 is similar to the proof of Theorem 4.

To prove Theorem 13 observe that Lemma 8.1 implies the following result about the continuous time system.

Lemma 12.1. *Given $\zeta > 0$ and $\varepsilon > 0$ there exists R such that for any two standard pairs $\ell_1 = (\gamma_1, \rho_1), \ell_2 = (\gamma_2, \rho_2)$ satisfying* (23)*,* (25) *and $|m_j| > R$ the following holds.*

Let $\bar{n} = |m_1|^{2(1+\zeta)}$. There exist positive constants $\bar{c}$ and $\bar{c}_\beta$, times $t_j(x) \leq \bar{L}\bar{n}$, probability measures $\bar{\nu}_1$ and $\bar{\nu}_2$, families of standard pairs and $\{\bar{\ell}_{\beta j}\}_{\beta, j=1,2}$ satisfying

$$\mathbb{E}_{\ell_j}(A \circ g_{t_j}) = \bar{c}\bar{\nu}_j(A) + \sum_\beta \bar{c}_\beta \mathbb{E}_{\bar{\ell}_{\beta j}}(A) \qquad j = 1, 2 \tag{32}$$

with $\bar{c} \geq 1 - \varepsilon$ and a measure preserving map $\bar{\pi} : (\gamma_1, \bar{\nu}_1) \to (\gamma_2, \bar{\nu}_2)$ such that for any $t > 0$ $d(g^{t+t_1}(x), g^{t+t_2}(\bar{\pi}x)) \leq C\theta^t$.

Now the proof of Theorem 13 is similar to the proof of Theorem 5. Namely it suffices to show that if x_j are distributed according to ℓ_j and $d(\ell_1, \ell_2), d(\ell_3, \ell_4) \gg 1$ then the distributions of

$$\pi(g_{tau(x_1,x_2)}x_1, g_{tau(x_1,x_2)}x_2) \text{ and}$$

$$\pi(g_{tau(x_3,x_4)}x_3, g_{tau(x_3,x_4)}x_4)$$

are close.

As in the proof of Theorem 5 we can assume that $d(\ell_1, \ell_2)$ and $d(\ell_3, \ell_4) \gg 1$ are comparable. By translation invariance we can assume also that ℓ_1 and ℓ_3 are close. Then we apply Lemma 12.1 to couple x_1 to x_3 and x_2 to x_4 and conclude as in the proof of Theorem 5.

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