A Limit Shape Theorem for Periodic Stochastic Dispersion

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Abstract

We consider the evolution of a connected set on the plane carried by a space periodic incompressible stochastic flow. While for almost every realization of the stochastic flow at time t most of the particles are at a distance of order $\sqrt{t}$ away from the origin, there is a measure zero set of points that escape to infinity at the linear rate. We study the set of points visited by the original set by time t and show that such a set, when scaled down by the factor of t, has a limiting nonrandom shape. © 2004 Wiley Periodicals, Inc.

1 Introduction

This paper deals with the long-time behavior of a passive scalar carried by an incompressible random flow. As has been demonstrated for a large class of stochastic flows with zero mean, under some mixing conditions on the flow, the displacement of a single particle is typically of order $\sqrt{t}$ for large t. In [8] we show that for almost every realization of the random flow, if one considers the image of an open set under the action of the flow, then its spatial distribution, scaled by the square root of time, converges weakly to a Gaussian distribution. On the other hand, it has been shown in the work of Cranston, Scheutzow, Steinsaltz, and Lisei [6, 7, 11, 14] that in any open set there are points that escape to infinity at a linear rate. In [9] we show that linear escape points form a set of full Hausdorff dimension. Denote the original set by Ω. One can think of Ω as an oil spill or a pollutant, say, on the surface of the ocean. The evolution of the set under the action of the flow will be denoted by Ω_t.

We shall study the set of "poisoned" points, that is, those visited by the image of Ω before time t

$$\mathcal{W}_t(\Omega) = \bigcup_{s \le t} \Omega_s \, .$$

As shown in [6, 7] the diameter of this set grows linearly in time almost surely. We shall be interested in its limit shape (scaled by t).

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Consider a stochastic flow of diffeomorphisms on $\mathbb{R}^2$ generated by a finite-dimensional Brownian motion

$$dx_t = \sum_{k=1}^{d} X_k(x_t) \circ d\theta_k(t) + X_0(x_t)dt \tag{1.1}$$

where $X_0, \dots, X_d$ are C^∞ smooth, divergence free, periodic vector fields and $\vec{\theta}(t) = (\theta_1(t), \dots, \theta_d(t))$ is a standard $\mathbb{R}^d$-valued Brownian motion with filtration $\mathcal{F}_t$. Let $f_{t,u}x$ be the solution at time u of the stochastic flow (1.1) with the initial data $x_t = x$.

We impose several assumptions on the vector fields $X_0, \dots, X_d$, which are stated in the next section (cf. [8]). All those, except the assumption of zero drift, are *nondegeneracy assumptions* and are satisfied for a generic set of vector fields $X_0, \dots, X_d$.

The main result of this paper is the following:

THEOREM 1.1 (Shape Theorem) *Let the original set* Ω *be bounded and contain a continuous curve with positive diameter. Under Assumptions* A *through* E *from Section* 1.1 *on the vector fields, there is a compact, convex, nonrandom set* $\mathcal{B}$*, independent of* Ω*, such that for any* $\varepsilon > 0$ *almost surely*

$$(1-\varepsilon)t\mathcal{B} \subset \mathcal{W}_t(\Omega) \subset (1+\varepsilon)t\mathcal{B} \tag{1.2}$$

for all sufficiently large t.

In [8] we prove that for a uniform initial measure on a curve, the image of the measure under the flow is asymptotically Gaussian. In Section 3 we use a result of this type, together with subadditivity arguments, to obtain a linear lower bound on the expected time for the image of the curve to reach a faraway point. We then show that this bound in turn implies the lower bound in (1.2) for the set $\mathcal{W}_t$.

The key element in the proof of the upper bound of (1.2) for $\mathcal{W}_t$ is to show that the set $\mathcal{W}_t$ for large t is almost independent of the original set (which, as will be demonstrated, can be taken to be a curve). In order to prove this, we show that given two bounded curves γ and γ', we can almost surely find a contour that contains γ' inside and that consists of a finite number of integer shifts of γ_t, and a finite number of stable manifolds of the stochastic flow (1.1) (whose length tends to zero as they evolve with the flow). In this way we see that if a point is visited by the image of γ', then its small neighborhood is earlier visited by the image of γ.

In Section 4 we describe the construction of the contour and provide the proof of the upper bound of (1.2). Section 2 contains necessary preliminaries. Some more technical estimates are collected in appendices.

1.1 Nondegeneracy Assumptions

In this section we formulate a set of assumptions on the vector fields $X_0, \dots, X_d$ that, in particular, imply the central limit theorem for measures carried by the flow (1.1) (see [8]). Such estimates are used in the proof of the shape theorem. Recall that $X_0, \dots, X_d$ are assumed to be periodic and divergence free. We shall assume that the period for all of the vector fields is equal to one.

Assumption A: *Strong Hörmander condition for* x_t. For all $x \in \mathbb{R}^2$ we have

$$\mathrm{Lie}(X_1, \dots, X_d)(x) = \mathbb{R}^2 ,$$

where $\mathrm{Lie}(X_1, \dots, X_d)(x)$ is the linear span of all possible Lie brackets of all orders formed out of $X_1, \dots, X_d$ at x. See Section 2.3 for consequences of the strong Hörmander condition for x_t.

Denote the diagonal in $\mathbb{T}^2 \times \mathbb{T}^2$ by

$$\Delta = \{(x^1, x^2) \in \mathbb{R}^2 \times \mathbb{R}^2 : x^1 = x^2 \ (\mathrm{mod}\, 1)\} .$$

Assumption B: *Strong Hörmander condition for the two-point motion*. The generator of the two-point motion $\{(x_t^1, x_t^2) : t > 0\}$ is nondegenerate away from the diagonal Δ, meaning that the Lie brackets made out of $(X_1(x^1), X_1(x^2))$, $\dots, (X_d(x^1), X_d(x^2))$ generate $\mathbb{R}^2 \times \mathbb{R}^2$.

To formulate the next assumption we need additional notation. Let $Dx_t : T_{x_0}\mathbb{R}^2 \to T_{x_t}\mathbb{R}^2$ be the linearization of x_t at t. We need the strong Hörmander condition for the process $\{(x_t, Dx_t) : t > 0\}$. Denote by TX_k the derivative of the vector field X_k thought of as the map on $T\mathbb{R}^2$ and by $S\mathbb{R}^2 = \{v \in T\mathbb{R}^2 : |v| = 1\}$ the unit tangent bundle on $\mathbb{R}^2$. If we denote by $\tilde{X}_k(v)$ the projection of $TX_k(v)$ onto $T_vS\mathbb{R}^2$, then the stochastic flow (1.1) on $\mathbb{R}^2$ induces a stochastic flow on the unit tangent bundle $S\mathbb{R}^2$ defined by the following equation:

$$d\tilde{x}_t = \sum_{k=1}^{d} \tilde{X}_k(\tilde{x}_t) \circ d\theta_k(t) + \tilde{X}_0(\tilde{x}_t)dt .$$

With this notation we have the following condition:

Assumption C: *Strong Hörmander condition for* (x_t, Dx_t). For all $v \in S\mathbb{R}^2$ we have

$$\mathrm{Lie}(\tilde{X}_1, \dots, \tilde{X}_d)(v) = T_vS\mathbb{R}^2 .$$

Let $L_{X_k}X_k(x)$ denote the derivative of X_k along X_k at the point x. Notice that $\frac{1}{2}\sum_{k=1}^{d} L_{X_k}X_k + X_0$ is the deterministic component of the stochastic flow (1.1) rewritten in Ito's form. Conditions A through C guarantee that the flow (1.1) has Lyapunov exponents and one of them is positive (see Section 2.5). We require that the flow have no deterministic drift, which is expressed by the following condition:

Assumption D: *Zero drift*.

$$\int_{\mathbb{T}^2} \left(\frac{1}{2}\sum_{k=1}^{d} L_{X_k}X_k + X_0\right)(x)dx = 0 . \tag{1.3}$$

We further require that

$$\int_{\mathbb{T}^2} X_k(x)dx = 0\,, \quad k = 1, \dots, d. \tag{1.4}$$

The last assumption is concerned with the geometry of the stream lines for one of the vector fields $X_1, \dots, X_d$. Fix a coordinate system on the 2-torus $\mathbb{T}^2 = \{x = (x_1, x_2) \bmod 1\}$. Since the vector fields have zero mean and are divergence free, there are periodic *stream* functions $H_1, \dots, H_d$ such that $X_k(x) = (-H'_{x_2}, H'_{x_1})$. We require the following:

Assumption E: *Morse condition on the critical points of* H_1. All of the critical points of H_1 are nondegenerate.

Functions with this property are called *Morse functions*. In Appendix E we show that a generic function has this property.

2 Background

In this section we collect some background information used throughout the paper.

2.1 Frostman Lemma

Given a probability measure ν, let $I_p(\nu)$ denote its p-energy

$$I_p(\nu) = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{d\nu(x)\, d\nu(y)}{|x-y|^p}\,. \tag{2.1}$$

Given a compact set $\Omega \in \mathbb{R}^2$, the q-Hausdorff measure $H^q(\Omega)$ of it is defined as follows: For any $\varepsilon > 0$ denote by U_ε the set of balls of radius at most ε covering Ω. Denote by R_ε the set of radii of balls from U_ε and let

$$H^q(\Omega) = \liminf_{\varepsilon \to 0} \inf_{U_\varepsilon} \sum_{r \in R_\varepsilon} r^q\,, \tag{2.2}$$

where the infimum is taken over all U_ε covers. We shall use the following fact from fractal geometry.

LEMMA 2.1 [12, theorem 8.8 and inequalities on p. 109] *Given positive q, p, m, and l with $q > p$ there exists a constant $J = J(q, p, m, l)$ such that if $\Omega \in \mathbb{R}^2$ is a set with* $\mathrm{diam}(\Omega) < l$ *and the q-Hausdorff measure $H^q(\Omega) \geq m$, then there is a measure ν on Ω of p-energy $I_p(\nu) \leq J$.*

2.2 Markov-Martingale Bound

The following estimate will be repeatedly used in the paper.

LEMMA 2.2 *Let* $\{\xi_j\}_{j\in\mathbb{Z}_+}$ *be a sequence of random variables such that*

$$\mathbb{E}(\xi_{j+1} \mid \xi_1 \dots \xi_j) \le 0\,,$$

and that for any m *the sequence* $\{\mathbb{E}|\xi_j|^m\}_j$ *is bounded by a constant* K_m*. Then for any* $\varepsilon > 0$ *there exists* $\kappa = \kappa(\varepsilon, m, K_m) > 0$ *such that for each* $n \in \mathbb{Z}_+$ *we have*

$$\mathbb{P}\left\{\sum_{j=1}^{n} \xi_j \ge \varepsilon n\right\} \le \kappa n^{-m}\,.$$

PROOF: Define a set of random variables $\{\zeta_j\} = \{\xi_j - \mathbb{E}(\xi_j \mid \xi_1 \cdots \xi_{j-1})\}_j$. Then $M_n = \sum_{j=1}^n \zeta_j$ is a martingale whose quadratic variation is equal to $\langle M\rangle_n = \sum_{j=1}^n \zeta_j^2$. By the martingale inequality

$$\mathbb{E}M_n^{2m} \le C_m \mathbb{E}\langle M\rangle_n^m \le C'_m n^m\,.$$

Therefore, by the Chebyshev inequality

$$\mathbb{P}\left\{\sum_{j=1}^{n} \zeta_j \ge \varepsilon n\right\} \le \mathbb{P}\left\{M_n^{2m} \ge (\varepsilon n)^{2m}\right\} \le \kappa n^{-m}\,.$$

Since $\xi_j \le \zeta_j$ the result also holds for the original sequence $\{\xi_j\}_j$. □

2.3 Positive Transition Density

Let vector fields $\{X_k\}_{k=0}^d$ be C^∞ smooth on a manifold M, and suppose they satisfy the strong Hörmander condition. For $t > 0$ let $p_t(x, dy)$ be time t transition probability for the process x_t defined in (1.1). Then by the Hörmander hypoellipticity principle

$$p(x, y, t) = \frac{dp_t(x, y)}{dy}$$

is a smooth function. By [10, theorem II.3] if M is compact then there exists a positive continuous function $c(t)$, $t > 0$, such that $p(x, y, t) \ge c(t)$. (See also [4, corollary 3.1]).

2.4 Closeness to the Deterministic Control

Let the vector fields $\mathcal{X} = \{X_k\}_{k=1}^d$ be C^∞ smooth on $\mathbb{T}^2$, and suppose they satisfy the strong Hörmander condition. Define L_k as follows. Let

$$L_1 = \{X_1 \cdots X_d\}$$

be the linear span of the vector fields. If L_{k-1} is already defined, let L_k be the union of L_{k-1} with the set of Lie brackets

$$L_k = L_{k-1} \cup \{[X, Y],\, X, Y \in L_{k-1}\}\,.$$

Denote $L = \bigcup_{k \in \mathbb{Z}_+} L_k$. By $\mathcal{X}$*-simple control* we mean a piecewise C^1 map $Z(t,x)$ from [0,1] to vector fields on $\mathbb{T}^2$ such that on each piece $Z(t,x) = v(t)Y(x)$ for some piecewise continuous function $v(t)$ on [0,1] and $Y \in L$. If Z is a simple control, let $\Phi(Z,t)$ denote the flow generated by $\dot{x} = Z(t,x)$.

The following is a slight generalization of [15]:

THEOREM 2.3 *Let $Z(t,x)$ be an $\mathcal{X}$-simple control. Given $\varepsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that for the stochastic flow* (1.1)

$$\mathbb{P}\left\{ \sup_{x\in\mathbb{T}^2, s\in[0,\delta_1]} \left| f_{0,s}x - \Phi\left(Z, \frac{s}{\delta_1}\right)x \right| < \varepsilon \right\} \geq \delta_2 .$$

The proof of this theorem is given in Appendix A.

2.5 Lyapunov Exponents

For measure-preserving stochastic flows with condition A, Lyapunov exponents λ_1 and λ_2 exist by the multiplicative ergodic theorem. Since our vector fields are divergence free, the sum of Lyapunov exponents $\lambda_1+\lambda_2$ is zero (see, e.g.,[3, p. 191]). Under conditions A through C the leading Lyapunov exponent is positive and almost surely does not depend on the initial vector. That is, there exists $\lambda_1 > 0$ such that for all x and v for almost all realizations of (1.1), we have

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log |d f_{0,t} x(v)| ,$$

where $f_{0,t}x$ is the solution at time t of the stochastic flow (1.1) with the initial data $x_0 = x$.

To see that λ_1 is positive, we note that theorem 6.8 of [2] states that under condition A the maximal Lyapunov exponent λ_1 can be zero only if one of the following two conditions is satisfied:

(a) there is a Riemannian metric $\mathbf{d}$ invariant with respect to all X_k, or
(b) there is a direction field $v(x)$ on $\mathbb{T}^2$ invariant with respect to all X_k.

However, (a) contradicts condition B. Indeed, (a) implies that all the Lie brackets of $\{(X_k(x^1),\ X_k(x^2))\}_k$ are tangent to the leaves of the foliation

$$\{(x^1, x^2) \in \mathbb{T}^2 \times \mathbb{T}^2 : \mathbf{d}(x^1, x^2) = \text{const}\}$$

and don't form the whole tangent space. On the other hand (b) contradicts condition C, since (b) implies that all the Lie brackets are tangent to the graph of v. This positivity of λ_1 is crucial for our approach.

2.6 No Superlinear Growth

We now state the lemma, proven in [7], which shows that γ_t cannot grow faster than linearly.

LEMMA 2.4 [7] *Let* γ *be the initial curve and*

$$\Phi_t = \sup_{x_0 \in \gamma} \sup_{0 \le s \le t} \|x_s - x_0\| .$$

(i) *There is a constant* C *such that almost surely*

$$\limsup_{t\to\infty} \frac{\Phi_t}{t} \le C .$$

(ii) *For any positive* r *and* α*, we have*

$$\sup_{t\ge 1} \mathbb{E}\left(\exp\left[\frac{r\Phi_t^2}{t^2 \max(1, \ln(\Phi_t/t))^{2+\alpha}}\right]\right) < \infty .$$

Note that due to periodicity of the vector fields X_k, both estimates are uniform with respect to γ. Indeed, we could at first consider γ coinciding with the boundary of the periodicity cell, from where the statement follows for all γ.

2.7 Central Limit Theorem

The next lemma, proven in section 5 of [8], describes the speed of propagation of a measure carried by the flow. Recall (2.1).

LEMMA 2.5 [8] *Let* ν *be a probability measure supported inside the ball* $B_R(0) \subset \mathbb{R}^2$ *whose* p *energy is bounded for some* $p > 0$*, that is,*

$$I_p(\nu) \le C_p < +\infty . \tag{2.3}$$

Let $f(x)$ *be a continuous, nonnegative function with compact support. Then there exists a nondegenerate* 2×2 *matrix* D *such that for any* $\rho, m > 0$ *there exists* $T = T(f, p, C_p, R, \rho, m)$ *such that for all* $t > T$

$$\mathbb{P}\left\{\left|\int_{\mathbb{R}^2} f\left(\frac{x_t}{\sqrt{t}}\right) d\nu - \bar{f}\right| > \rho\right\} \le t^{-m} , \tag{2.4}$$

where $\bar{f}$ *denotes the integral of* f *with respect to the Gaussian measure with zero mean and variance* D.

PROOF: The last inequality of section 7 in [8] establishes (2.4) for functions of the form $f(x) = \exp(i\xi x)$. Also, lemma 12 of [8] shows that there exists K such that for all m we have

$$\mathbb{P}\left\{\nu\left[\left(\frac{x_t}{\sqrt{t}}\right)^2 > K\right]\right\} \le C_m t^{-m} .$$

Let $\tilde{R} = 2\sqrt{\sup|f|K\rho}$. Then with probability at least $1 - C_m t^{-m}$ we have

$$\int_{|x_t| > \tilde{R}} f\left(\frac{x_t}{\sqrt{t}}\right) d\nu \le \frac{\rho}{4} .$$

We can uniformly approximate f on the ball $B_{\tilde{R}}(0)$ by a trigonometric polynomial, which implies the result. □

We shall use the following important consequence of Lemma 2.5. We shall call a curve *long* if its diameter is bounded and greater than or equal to 1.

COROLLARY 2.6 *Given $\varepsilon > 0$ and an integer $m > 0$ there exist C_m and T such that if γ is a curve with* $\operatorname{diam}(\gamma) \geq \varepsilon$ *then*

$$\mathbb{P}\{\gamma_t \text{ is long for all } t \geq T\} \geq 1 - C_m T^{-m} .$$

PROOF: The condition $\operatorname{diam}(\gamma) \geq \varepsilon$ implies that the 1-Hausdorff measure $H^1(\gamma) \geq \varepsilon$. Hence by the Frostman lemma there is a constant $a > 0$, independent of γ, and a measure ν supported on γ such that $I_{1/2}(\nu) \leq a$. Take two nonnegative functions f_1 and f_2 with disjoint, compact, nonempty supports. By Lemma 2.5 for all m and $N = N_m$ we have

$$\mathbb{P}\{\gamma_n \cap \sqrt{n}\ \operatorname{supp}(f_j) \neq \varnothing \text{ for all } n \in \mathbb{N}, n \geq N\} \geq 1 - \text{const}\, N^{-m} ,$$

that is, except for the set of small probability, $\operatorname{diam}(\gamma_n) \geq \text{const}\sqrt{n}$ at integer moments of time. On the other hand, by Lemma 2.4 for all m

$$\mathbb{P}\{\text{there is } t \in [n, n+1] \text{ such that } \mathcal{R}(\gamma_t, \gamma_n) \geq n^{1/4}\} \leq \text{const}\, n^{-m} ,$$

where $\mathcal{R}(\gamma_t, \gamma_n) = \sup_{y \in \gamma_t} \inf_{x \in \gamma_n} \operatorname{dist}(x, y)$. Combining the last two inequalities, we obtain Corollary 2.6. □

3 Lower Bound

3.1 Linear Growth and an Estimate from Below

Let the initial set be a curve $\gamma \subset \mathbb{R}^2$, and let A be a faraway point in the plane. We shall estimate the tail of the probability distribution of the time it takes for the curve to reach an R-neighborhood of A in terms of the distance between γ and A (the constant R will be selected later).

By Corollary 2.6 we may assume without loss of generality that the original curve is long. Given a long curve γ and a point A, we define $\tau^R(\gamma, A)$ to be the first moment of time when the image of γ reaches the R-neighborhood of A, and at the same time the image of γ is long, that is

$$\tau^R(\gamma, A) = \inf\{t > 0 : \operatorname{dist}(\gamma_t, A) \leq R,\ \operatorname{diam}(\gamma_t) \geq 1\} . \tag{3.1}$$

PROPOSITION 3.1 *Consider a long curve $\gamma \in \mathbb{R}^2$, and a point $A \in \mathbb{R}^2$. Let $d =$* $\max\{1, \operatorname{dist}(A, \gamma)\}$. *There is a constant $R > 0$, and for any positive integer m there is $C_m > 0$, independent of γ, A, and d, for which*

$$\mathbb{P}\left\{\tau^R(\gamma, A) > C_m \beta\, d\right\} \leq C_m \beta^{-m}\, d^{-m} \quad \text{for any } \beta > 1 . \tag{3.2}$$

The proof of Proposition 3.1 will rely on Lemma 3.2 stated below.

Choose $A_0 \in \gamma$. Now, given a triplet (A_0, γ_0, t_0), where $\gamma_0 = \gamma$ is a long curve in $\mathbb{R}^2$, A_0 is a point on γ, and $t_0 = 0$ is the initial time, we define inductively the sequence $\{(A_j, \gamma_j, t_j)\}$ as follows: Suppose that (A_j, γ_j, t_j) is defined so that

- γ_j is a connected interval of the image of γ_{j-1}, i.e., $\gamma_j \subset f_{t_{j-1}, t_j} \gamma_{j-1}$,

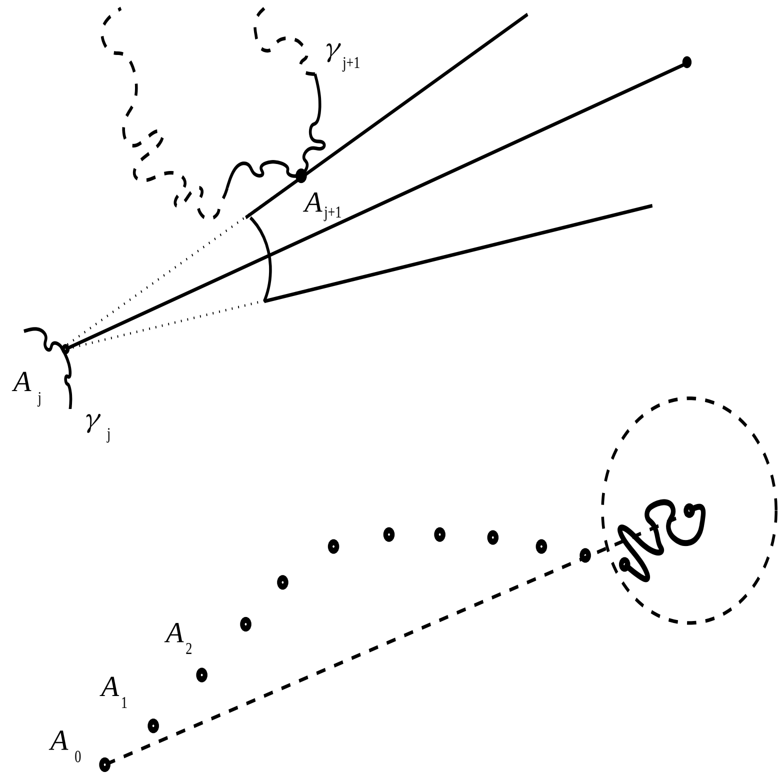

FIGURE 3.1

- $A_j \in \gamma_j$, and
- γ_j is long.

Given $\alpha \in (0, \frac{\pi}{2})$ define the truncated α-cone

$$K_j(\alpha) = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, A_j) \geq 1 \text{ and } \angle(xA_jA) \leq \alpha\}\,, \tag{3.3}$$

where $\angle(xA_jA)$ is the angle between the segments $[A_j, x]$ and $[A_j, A]$. See Figure 3.1.

Let t_{j+1} be the first moment such that

- $t_{j+1} - t_j \geq 1$,
- $\operatorname{diam}(f_{t_j,t_{j+1}}\gamma_j) \geq 1$, and
- $f_{t_j,t_{j+1}}\gamma_j \bigcap K_j(\alpha) \neq \varnothing$.

Let A_{j+1} be an arbitrary point in $\gamma_j(t_{j+1}) \cap K_j(\alpha)$, let $B_R(A_{j+1})$ denote the closed R-ball around A_{j+1}, and let γ_{j+1} be a long curve that satisfies

$$A_{j+1} \in \gamma_{j+1} \subseteq f_{t_j,t_{j+1}}\gamma_j \cap B_1(A_{j+1})\,.$$

LEMMA 3.2 *Fix* $0 < \alpha < \frac{\pi}{2}$. *For any positive integer m we have*

$$\mathbb{E}\big((t_{j+1} - t_j)^m\big) \mid \mathcal{F}_{t_j}) < C_m\,.$$

PROOF: It is sufficient to prove that $\mathbb{E}(t_1^m) < C_m$ with C_m independent of the original long curve γ_0 and of the point A. Without loss of generality we may assume that the original curve is contained in a ball of radius 2 centered around the origin. Note that there is a finite set of functions with compact supports such that for any cone $K_0(\alpha)$, defined in (3.3), with the vertex inside $B_2(0)$ there is a function f from this set for which r supp $f \subset K_0(\alpha)$ for all $r \geq 1$.

Since γ_0 is long, the Frostman lemma (Section 2.1) implies that there exists a probability measure ν on γ_0 whose $\frac{1}{2}$-energy is bounded; therefore Lemma 2.5 can be applied. Corollary 2.6 implies that for large t and for each of the functions from the finite set we have

$$\mathbb{P}\{\gamma_t \cap \{\sqrt{t}\ \text{supp}\ f\} = \varnothing\} \leq t^{-m}\,.$$

Since for $t \geq 1$ for one of the functions we have that $\{\sqrt{t}\,\text{supp}\ f\} \subset K_0(\alpha)$, we get that for $t \geq T$

$$\mathbb{P}\{\gamma_t \cap K_0(\alpha) = \varnothing\} \leq t^{-m}\,.$$

Corollary 2.6 implies that for large t

$$\mathbb{P}\{\text{diam}(\gamma_t) < 1\} \leq t^{-m}\,.$$

Since m is arbitrary, this implies the required result. □

PROOF OF PROPOSITION 3.1: Let $r_j = \text{dist}(A_j, A)$. There exist positive constants R and K such that if $r_0 > R$, then $\mathbb{E}(r_1 - r_0) \leq -K$. Indeed, due to lemmas 2.4 and 3.2, the tail of the distribution of $\text{dist}(A_0, A_1)$ decays faster than any power, uniformly in A_0, γ_0. By selecting R large enough and K small enough, we can assure that $\text{dist}(A_1, A) < \text{dist}(A_0, A) - 2K$ with probability arbitrarily close to 1. The contribution to the expectation from the complementary event is estimated using the decay of the tail of the distribution of $\text{dist}(A_0, A_1)$.

Let $\sigma \in \mathbb{N}$ be the first moment when $r_j \leq R$, $\sigma = \min_j\{r_j \leq R\}$. Then

$$S_j = r_{\min(j,\sigma)} + K \min(j,\sigma) \tag{3.4}$$

is a supermartingale. Notice that by Lemmas 2.4 and 3.2 the sequence $\xi_j = S_j - S_{j-1}$ satisfies the assumptions of Lemma 2.2 with the constant K_m, and K_m can be chosen independently of the distance d. Therefore, there is $\kappa > 0$ such that

$$\mathbb{P}\left\{r_{\min(j,\sigma)} + K\min(j,\sigma) \geq \frac{Kj}{2} + d\right\} \leq \kappa j^{-m} \quad \text{for all } j \geq 1.$$

Take an arbitrary $\beta > 1$ and let $j_0 = [\frac{2d}{K}\beta] + 1$. Since $r_{\min(j_0,\sigma)}$ is nonnegative, the event $\{\sigma \geq j_0\}$ is contained in the event

$$\left\{r_{\min(j_0,\sigma)} + K\min(j_0,\sigma) \geq \frac{Kj_0}{2} + d\right\}.$$

Thus,

$$\mathbb{P}\{\sigma \geq j_0\} \leq \kappa j_0^{-m} .$$

We conclude that for some $\tilde{C}_m > 0$

$$\mathbb{P}\{\sigma \geq \tilde{C}_m \beta d\} \leq \tilde{C}_m \beta^{-m} d^{-m} \quad \text{for } \beta, d > 1. \tag{3.5}$$

This inequality is different from (3.2) in that (3.5) provides an estimate for the number of steps, rather than time, needed to reach an R-neighborhood of A. By Lemma 3.2 we can apply Lemma 2.2 with $\xi_n = t_{n+1} - t_n - C$ for some positive C to obtain that for any m there exists κ such that

$$\mathbb{P}\{t_n \geq 2Cn\} \leq \kappa n^{-m} .$$

This, together with (3.5), implies the statement of the proposition. □

Let $\mathcal{W}_t^R(\gamma)$ be the R-neighborhood of $\mathcal{W}_t(\gamma)$, that is, the set of points whose R-neighborhood is visited by the image of the original set before time t,

$$\mathcal{W}_t^R(\gamma) = \{x \in \mathbb{R}^2 : \text{dist}(x, \gamma_s) \leq R \text{ for some } s \leq t\} .$$

COROLLARY 3.3 *There exist positive constants c and R such that almost surely for t large enough $\mathcal{W}_t^R(\gamma)$ contains the ball of radius ct centered at the origin, i.e., $B_{ct}(0) \subset \mathcal{W}_t^R(\gamma)$ for large t.*

PROOF: Consider a covering of $B_{ct}(0)$ by balls of radius $R/2$. By Proposition 3.1 for each of the balls of radius $R/2$, the probability that it is not visited by the curve by time t decays faster than any power of t, provided that c is small enough and R is large enough. On the other hand, for each c and R the number of balls needed to cover $B_{ct}(0)$ grows like t^2 times a constant. Therefore, the probability that the R-neighborhood of some point in $B_{ct}(0)$ is not visited by the curve before time t decays faster than any power of t. The corollary follows by the Borel-Cantelli lemma. □

From now on we fix R for which Proposition 3.1 and Corollary 3.3 hold.

Note that the bounds we obtained in the proof of Proposition 3.1 are uniform over all long curves. Let us employ this fact in the following corollary. Let $\mathcal{C}_R$ be the family of long curves that lie completely inside $B_{2R}(0)$ (we may assume that $R > 1$).

COROLLARY 3.4 *The family of stopping times, defined in* (3.1),

$$\left\{ \frac{\tau^R(\gamma, tv)}{t} \right\}_{t \geq 1, \|v\|=1, \gamma \in \mathcal{C}_R}$$

is uniformly integrable.

3.2 Stable Norm

We shall now use the asymptotics of the stopping time τ^R defined in (3.1) in order to define the limiting shape $\mathcal{B} \subset \mathbb{R}^2$. Recall that τ^R is the time it takes a curve to reach the R-neighborhood of a faraway point. Consider

$$|v|^R = \sup_{\gamma \in \mathcal{C}_R} \mathbb{E}\tau^R(\gamma, v)\,.$$

The stationarity of the underlying Brownian motion and the periodicity of the vector fields imply that

$$\mathbb{E}\tau^{2R}(\gamma, (t_1+t_2)v) \le \mathbb{E}\tau^R(\gamma, t_1 v) + \mathbb{E}\tau^R(\gamma_1, t_2 v)\,,$$

where $\gamma_1 \in \mathcal{C}_R$ is some integer translation of a part of $f_{0,\tau(\gamma,t_1 v)}\gamma$. By Proposition 3.1

$$\mathbb{E}\tau^R(\gamma, (t_1+t_2)v) \le \mathbb{E}\tau^{2R}(\gamma, (t_1+t_2)v) + C$$

for some $C > 0$. It follows that the function $|tv|^R + C$ is subadditive. Let

$$\|v\|^R = \lim_{t\to\infty} \frac{|tv|^R}{t}\,. \tag{3.6}$$

Similarly, for $0 \le s \le 1$,

$$|t(sv_1 + (1-s)v_2)|^R \le |tsv_1|^R + |t(1-s)v_2|^R + C\,,$$

so

$$\|sv_1 + (1-s)v_2\|^R \le s\|v_1\|^R + (1-s)\|v_2\|^R\,.$$

Let $\mathcal{B} = \{v \in \mathbb{R}^2 : \|v\|^R \le 1\}$. By the remarks above $\mathcal{B}$ is convex. By Lemma 2.4 $\mathcal{B}$ has nonempty interior. By Corollary 3.4 $\mathcal{B}$ is compact. It will be shown that the norm $\|v\|^R$ and the set $\mathcal{B}$ are independent of R.

LEMMA 3.5 *For any curve $\gamma \in \mathcal{C}_R$, any $\varepsilon > 0$, and almost every realization of the Brownian motion $\vec\theta(t)$ there exists $T = T(\gamma, \varepsilon, \vec\theta(t)) > 0$ such that $(1-\varepsilon)t\mathcal{B} \subset \mathcal{W}_t^R(\gamma)$ for $t \ge T$.*

PROOF: It suffices to show that for all v with $\|v\|^R \le 1$ and any m there is C_m such that

$$\mathbb{P}\{\tau^R(\gamma, tv) \ge (1+\varepsilon)t\} \le C_m t^{-m}\,. \tag{3.7}$$

All the estimates below are uniform in v such that $\|v\|^R = 1$. By the definition of $\|v\|^R$ there exists t_0 such that

$$\mathbb{E}\tau^R(\gamma, tv) \le t\left(1 + \frac{\varepsilon}{3}\right) \tag{3.8}$$

for any $t \ge t_0$ and $\gamma \in \mathcal{C}_R$. Define the stopping time τ_1^R as

$$\tau_1^R = \inf\{t > 0 : \gamma_t \cap B_R(t_0 v) \ne \varnothing;\ \operatorname{diam}(\gamma_t) \ge 1\}\,.$$

Recall that $B_R(t_0 v)$ is the R-ball centered at $t_0 v$. Let $\gamma^{(1)}$ be a long part of $\gamma_{\tau_1^R}$ contained in $B_{2R}(t_0 v)$ that has a nonempty intersection with $B_R(t_0 v)$. Similarly, we define τ_2^R to be the first time following τ_1^R when the image of $\gamma^{(1)}$ is long and intersects $B_R(2t_0v)$. Let $\gamma^{(2)}$ be a long part inside the image $f_{\tau_1^R,\tau_2^R}\gamma^{(1)}$ of $\gamma^{(1)}$, and so on. We have therefore constructed a sequence of stopping times such that

$$\tau^R(\gamma, nt_0v) \le \sum_{j=1}^{n} \left(\tau_j^R - \tau_{j-1}^R\right).$$

By (3.8), due to the periodicity of the underlying vector fields, for large enough t_0 we have

$$\mathbb{E}(\tau_j^R - \tau_{j-1}^R) \le t_0\left(1 + \frac{2\varepsilon}{3}\right).$$

Now the result follows by Lemma 2.2 and Proposition 3.1. ☐

Now we prove that Lemma 3.5 remains valid even when the R-neighborhood of $\mathcal{W}_t(\gamma)$ is replaced by $\mathcal{W}_t(\gamma)$ itself.

THEOREM 3.6 *For any $\gamma \in \mathcal{C}_R$ and any $\varepsilon > 0$ we have almost surely $(1-\varepsilon)t\mathcal{B} \subset \mathcal{W}_t(\gamma)$ for large enough t.*

This theorem is a consequence of Lemma 3.5 and the fact that when a long curve reaches an R-neighborhood of a point, then the distribution of the time it takes for the curve to sweep the entire neighborhood has a fast decreasing tail. Thus Theorem 3.6 follows from the standard Borel-Cantelli arguments and the following sweeping lemma:

LEMMA 3.7 *Let γ be a long curve such that* $\mathrm{dist}(\gamma, A) \le R$. *Let*

$$\sigma = \inf\left\{t > 0 : B_R(A) \subset \bigcup_{s \le t} \gamma_s\right\}.$$

Then for any $m > 0$ and some C_m that does not depend on γ we have

$$\mathbb{P}\{\sigma > t\} \le C_m t^{-m}. \tag{3.9}$$

The proof of this lemma is the subject of Appendix B.

4 Upper Bound

4.1 Stable Manifold

We first recall some properties of stable manifolds. Recall that $f_{t,u}x$ be the solution at time u of the stochastic flow (1.1) with the initial data $x_t = x$. Recall that $\lambda_1 > 0$ is a maximal Lyapunov exponent, as discussed in Section 2.5. Consequently $\lambda_2 = -\lambda_1 < 0$. Let $0 < \bar{\lambda}_1 < \lambda_1$. Then, by the stable manifold theorem [5, sec. 2.2], for every t and every x almost surely the set

$$W^s(x,t) = \left\{y \in \mathbb{R}^2 : d(f_{t,u}y,\ f_{t,u}x) \le C(y)e^{-\bar{\lambda}_1(u-t)} \text{ for some } C(y) \text{ and } u \ge t\right\}$$

is a smooth curve passing through x.

4.2 Estimates in Probability

We first establish the asymptotics of the expectation of $\tau^R(\gamma, tv)$.

THEOREM 4.1 *The limit*

$$\lim_{t\to\infty} \frac{\mathbb{E}\tau^R(\gamma, tv)}{t} = \|v\|^R$$

is uniform in $\gamma \in \mathcal{C}_R$, *where* $\mathcal{C}_R$ *is defined before Corollary* 3.4.

The proof of Theorem 4.1 is based on the following proposition (proven after the theorem). Let $W^{s,L}$ be a connected part of the stable manifold $W^s(x,0)$ of a point x such that it satisfies $\sup_{t\geq 0} \mathrm{diam}(f_{0,t} W^{s,L}) \leq L$.

PROPOSITION 4.2 *For any* $\varepsilon > 0$ *there is* $L = L(\varepsilon)$ *such that for any two long curves* $\gamma_1, \gamma_2 \in \mathcal{C}_{2R}$

$$\mathbb{P}\{\textit{there exists } W^{s,L} \textit{ connecting } \gamma_1 \textit{ and } \gamma_2\} > 1 - \varepsilon\,.$$

PROOF OF THEOREM 4.1: Denote by S the boundary of the square, $S = \partial[-2R, 2R]^2$. Since $\tau^R(S, tv) \leq \tau^R(\gamma, tv)$ for $\gamma \in \mathcal{C}_R$, by Corollary 3.4 the family of random variables

$$\left\{\frac{\tau^R(\gamma, tv) - \tau^R(S, tv)}{t}\right\}_{t\geq 1, \|v\|=1, \gamma\in\mathcal{C}_R}$$

is uniformly integrable. We shall demonstrate that for any $\varepsilon > 0$

$$\mathbb{P}\left\{\frac{\tau^R(\gamma, tv) - \tau^R(S, tv)}{t} > \varepsilon\right\} \to 0 \quad \text{uniformly in } \|v\| = 1\,, \quad \gamma \in \mathcal{C}_R\,. \tag{4.1}$$

From the uniform integrability and (4.1), it then follows that

$$\frac{\mathbb{E}\tau^R(\gamma, tv) - \mathbb{E}\tau^R(S, tv)}{t} \to 0 \quad \text{uniformly in } \|v\| = 1\,, \quad \gamma \in \mathcal{C}_R\,,$$

which implies the statement of the theorem. It remains to prove (4.1).

Given $\varepsilon > 0$ we select $L = L(\varepsilon/3)$ according to Proposition 4.2. We set γ_1 equal to γ and γ_2 equal to a translation of γ by a unit vector in either the horizontal or vertical direction. In either case we can apply Proposition 4.2. Besides, due to the periodicity of the flow, we can apply Proposition 4.2 to any integer translation of the pair (γ_1, γ_2). We obtain that with probability not less than $1 - 2\varepsilon/3$ there exists a contour Γ, which contains S and is contained in $[-10R - L, 10R + L]^2$. The contour Γ consists of a finite number of integer translations of γ and a finite number of integer translations of two stable manifolds $W_1^{s,L}$ and $W_2^{s,L}$. The former manifold connects γ with its horizontal translation, and the latter one connects γ with its vertical translation.

Since $f_{0,t}\Gamma$ consists of integer translations of $f_{0,t}\gamma$ at most distance $30R + 3L$ away from each other and stable manifolds have length no greater than L, we have

$$\tau^{31R+4L}(\gamma, tv) \leq \tau^R(\Gamma, tv)\,.$$

Since Γ contains S, we have

$$\mathbb{P}\{\Gamma \text{ as above exists and } \tau^R(\Gamma, tv) \le \tau^R(S, tv)\} \ge 1 - \frac{2\varepsilon}{3}\,.$$

By Proposition 3.1 for sufficiently large t we have

$$\mathbb{P}\{\tau^R(\gamma, tv) - \tau^{31R+3L}(\gamma, tv) > \varepsilon t\} \le \frac{\varepsilon}{3}\,.$$

Combining the last three inequalities, we obtain

$$\mathbb{P}\left\{\frac{\tau^R(\gamma, tv) - \tau^R(S, tv)}{t} > \varepsilon\right\} \le \varepsilon\,,$$

which implies (4.1). This completes the proof of Theorem 4.1. □

PROOF OF PROPOSITION 4.2: We need to introduce some notation. Figure D.1 of Appendix D can be helpful here. Recall that we denote the stream function of X_1 by H_1. The streamlines of X_1 are level sets of H_1. See Appendix E for properties of level sets of functions satisfying condition E. Any regular closed level set γ_0 of H_1 on the torus has a neighborhood where we can define action-angle coordinates $(I, \phi) \in [0, 1] \times \mathbb{S}^1$ such that the dynamics under the flow X_1 is described by $\dot\phi = \omega(I),\ \dot I = 0$. Let p be a maximum point of H_1. By assumption E a small neighborhood of p consists of closed level sets, so we can introduce action-angle coordinates. Let U be the maximal neighborhood of p where action-angle coordinates can be introduced. Then ∂U contains saddle critical points (or point) of H_1. Observe that all level sets in U are homotopic to a point (one such homotopy is obtained by moving along the integral curves of ∇H_1). So the level sets lift to closed level sets on $\mathbb{R}^2$. Abusing notation, we will denote the lifts of U and ∂U to the plane by the same letters.

Let us fix a point $x \in \partial U$, which is not a saddle point. Let us consider a cone $K^x = \{y : \|y - x\| \le a;\ (y - x, n) \ge b\|y - x\|\}$, where n is the unit inward normal at x, and $a, b > 0$ are constants. Let K_1^x and K_2^x be the two sides of the cone, that is, the points where $(y - x, n) = b\|y - x\|$, and let K_3^x be the remaining part of the boundary of the cone, where $\|y - x\| = a$.

Let $W_a^s(x, t)$ be the connected component of $W^s(x, t) \cap B_a(x)$ containing x. In Appendix C we prove the following:

LEMMA 4.3 *If $a = a(\varepsilon)$ and $b = b(\varepsilon)$ are small enough, and $L^0(\varepsilon)$ is large enough, then each of the following events has probability at least* $1 - \varepsilon/10$:

$$A^1 = \left\{W_a^s(x, 0) \cap \partial K^x \subseteq \{x\} \cup K_3^x,\ W_a^s(x, 0) \cap K_3^x \ne \varnothing\right\}, \tag{4.2}$$
$$A^2 = \left\{\sup_{u \ge 0} \operatorname{diam}(f_{0,u} W_a^s(x, 0)) \le L^0(\varepsilon)\right\}.$$

Due to stationarity in t of the flow, we also have

$$\mathbb{P}\{A_t^i\} \ge 1 - \frac{\varepsilon}{10}\,, \quad i = 1, 2, \tag{4.3}$$

where

$$A_t^1 = \left\{ W_a^s(x,t) \cap \partial K^x \subseteq \{x\} \cup K_3^x,\ W_a^s(x,t) \cap K_3^x \neq \varnothing \right\}$$
$$A_t^2 = \left\{ \sup_{u \geq t} \operatorname{diam}(f_{t,u} W_a^s(x,t)) \leq L^0(\varepsilon) \right\}.$$

In Appendix D we prove the following statement:

LEMMA 4.4 *There is $T(\varepsilon) > 0$ such that for any pair of curves $\gamma_1, \gamma_2 \in \mathcal{C}_{2R}$ with probability at least $1 - \varepsilon/10$ there is $t \leq T(\varepsilon)$ such that both $f_{0,t}\gamma_1$ and $f_{0,t}\gamma_2$ contain connected subsets $\overline{\gamma_1}$ and $\overline{\gamma_2}$, respectively, which belong to the cone K^x and such that*

$$\overline{\gamma_i} \cap K_1^x \neq \varnothing \quad \textit{and} \quad \overline{\gamma_i} \cap K_2^x \neq \varnothing\,, \quad i = 1, 2.$$

Assume this lemma is proven. Due to (4.3), with probability at least $1 - \varepsilon/2$ there is some $t \leq T(\varepsilon)$ such that $f_{0,t}\gamma_1$ and $f_{0,t}\gamma_2$ both intersect the same connected set $W_a^s(x,t)$ and the events A_t^i hold for $i = 1, 2$.

Note that the preimage under $f_{0,t}$ of $W_a^s(x,t)$ is a part of a stable manifold. Since $T(\varepsilon)$ is finite,

$$\mathbb{P}\left\{ \sup_{0 \leq u \leq t} \operatorname{diam}(f_{0,u} f_{0,t}^{-1} W_a^s(x,t)) > L(\varepsilon) \right\} \leq \frac{\varepsilon}{2}$$

for large enough $L(\varepsilon)$. If necessary, we can make $L(\varepsilon)$ yet larger to satisfy $L(\varepsilon) \geq L^0(\varepsilon)$. Therefore, with probability at least $1 - \varepsilon$ the curves γ_1 and γ_2 are connected by a part of a stable manifold $W^{s,L(\varepsilon)}$ such that

$$\sup_{u \geq 0} \operatorname{diam}(f_{0,u} W^{s,L(\varepsilon)}) \leq L(\varepsilon)\,,$$

which completes the proof of Proposition 4.2. □

COROLLARY 4.5 *For any curve $\gamma \in \mathcal{C}_R$ we have $\lim_{t\to\infty} \tau^R(\gamma, tv)/t = \|v\|^R$ in probability.*

PROOF: By Corollary 3.4 for any curve $\gamma \in \mathcal{C}_R$ the family of measures on $\mathbb{R}$ induced by $\{\tau^R(\gamma, tv)/t\}_{t\geq 1}$ is tight. Let ν_γ be a limit distribution of this family. On one hand, Lemma 3.5 implies that supp $\nu_\gamma \subset [0, \|v\|^R]$. On the other hand, by Theorem 4.1 we have $\int s\, d\nu_\gamma(s) = \|v\|^R$. Thus $\nu_\gamma = \delta_{\|v\|^R}$. □

COROLLARY 4.6 *For any curve $\gamma \in \mathcal{C}_R$ and any $\varepsilon > 0$ we have*

$$\lim_{t\to\infty} \mathbb{P}\left\{ \mathcal{W}_t^R(\gamma) \subset (1+\varepsilon)t\mathcal{B} \right\} = 1\,.$$

PROOF: Let Ω_t be the following event:

$$\Omega_t = \left\{ \tau^R(\gamma, tv) < \frac{t}{1+\varepsilon} \text{ for some } v \text{ with } \|v\|^R = 1 \right\}.$$

We need to show $\mathbb{P}\{\Omega_t\} \to 0$ as $t \to +\infty$. Take $0 < \delta \ll \varepsilon$. For each $t > 0$ let $\{v_j\}$ be a δ-net in $\partial\mathcal{B}$, and let Ω_t^δ be the following event:

$$\Omega_t^\delta = \left\{ \tau^R(\gamma, tv_j) < \frac{t}{1+\varepsilon/2} \text{ for some } j \right\}.$$

Note that $\mathbb{P}\{\Omega_t^\delta\} \to 0$ by Corollary 4.5. If for some v with $\|v\| = 1$ at time $\tau^R(\gamma, tv) < t/(1+\varepsilon)$ the curve γ gets to tv, then the probability that γ hits tv_j with v_j one of the closest to v before $t/(1+\varepsilon/2)$ is close to 1 by Proposition 3.1. More exactly, for any m for sufficiently small δ, we have

$$\mathbb{P}\{\Omega_t^\delta \mid \Omega_t\} \geq 1 - C_m t^{-m}.$$

Therefore, $\mathbb{P}\{\Omega_t\} \to 0$ as $t \to \infty$, which is required. This completes the proof. □

4.3 Curve-to-Line Passage Time

As the reader will see in this section, we essentially use periodicity of the flow (1.1). Given a curve γ and a line l in the plane, we define $\tau^R(\gamma, l)$ to be the stopping time when the image of γ reaches the R-neighborhood of l and the image of γ is long. As in Section 3.2 we define

$$|l|^R = \sup_{\gamma \in \mathcal{C}_R} \mathbb{E}\tau^R(\gamma, l)$$

and, provided that the following limit exists, we define

$$\|l\|^R = \lim_{t\to\infty} \frac{|tl|^R}{t}. \tag{4.4}$$

The following results for $\|l\|^R$ can be proven exactly like the corresponding results (formula (3.6) and Corollary 4.5) for $\|v\|^R$.

LEMMA 4.7

(i) *The limit in* (4.4) *exists and, therefore,* $\|l\|^R$ *is well-defined.*
(ii) *For any* $\gamma \in \mathcal{C}_R$ *we have* $\lim_{t\to\infty} \tau^R(\gamma, tl)/t = \|l\|^R$ *in probability.*

The next lemma relates $\|l\|^R$ to the norm $\|v\|^R$.

LEMMA 4.8 *For any line l in the plane,*

$$\|l\|^R = \inf_{v \in l} \|v\|^R.$$

PROOF: Corollary 4.6 along with Lemma 4.7(ii) implies that $\|l\|^R \leq \inf_{v\in l} \|v\|^R$. From Lemmas 3.5 and 4.7(ii) it follows that $\|l\|^R \geq \inf_{v\in l} \|v\|^R$, which completes the proof. □

Provided that the following limit exists, we define:

$$\|l\|_*^R = \lim_{t\to\infty} \frac{\mathbb{E}(\tau^R(l', tl))}{t}$$

where l' is the line parallel to l passing through origin. Note that $\tau^R(l', tl)$ is defined as the first instance when the image of l' reaches an R-neighborhood of tl, the same way as $\tau^R(\gamma, tl)$ was defined for a compact γ.

LEMMA 4.9 *For any line l on the plane we have the equality*

$$\|l\|_*^R = \|l\|^R .$$

PROOF: Let us cover the line l' by the union of fundamental domains (unit squares whose vertices have integer coordinates) in such a way that every square has a nonempty intersection with l'. Let Π be the boundary of such a union of fundamental domains. Since Π lies in between two integer shifts of l' of distance not greater than 2, Proposition 3.1 and the periodicity of the flow imply that as $t \to \infty$

$$\mathbb{E}\frac{\tau^R(l', tl) - \tau^R(\Pi, tl)}{t} \to 0 .$$

Let S be the boundary of a fundamental domain that contains the origin. Again, Proposition 3.1 and the periodicity of the flow imply

$$\mathbb{E}\frac{\tau^R(S, tl) - \tau^R(\Pi, tl)}{t} \to 0 .$$

Subtracting one from the other, we get

$$\mathbb{E}\frac{\tau^R(l', tl) - \tau^R(S, tl)}{t} \to 0 .$$

Pass to the limit as $t \to \infty$. Due to Lemma 4.7(i) we get the required statement. □

4.4 Almost Sure Convergence

LEMMA 4.10 *For any curve $\gamma \in \mathcal{C}_R$, any $\varepsilon > 0$, and almost every realization of the Brownian motion $\vec{\theta}(t)$ there exists $T = T(\gamma, \varepsilon, \vec{\theta}(t)) > 0$ such that $\mathcal{W}_t^R(\gamma) \subset (1+\varepsilon)t\mathcal{B}$ for $t \geq T$.*

PROOF: The proof is somewhat analogous to the proof of Corollary 4.6. Choose small $0 < \delta \ll \varepsilon$ and let $\{v_j\}$ be an δ-net on $\partial\mathcal{B}$. Let $\mathcal{B}_\delta$ be the region bounded by support lines of $\mathcal{B}$ passing through $\{v_j\}$. In other words, we consider a polygon with side of length of order δ superscribed around $\mathcal{B}$. Since δ is small, it suffices to prove that almost surely $\mathcal{W}_t^R(\gamma) \subset (1+\varepsilon/2)t\mathcal{B}_\delta$ for large t. This inclusion follows from Lemma 4.8 if for each of the supporting lines l of $\mathcal{B}$ we show that almost surely the following inequality holds for sufficiently large t:

$$\tau^R(\gamma, tl) \geq \left(1 - \frac{\varepsilon}{4}\right) t\|l\|^R . \tag{4.5}$$

Let t^* be such that

$$\mathbb{E}\tau^R(l', tl) > \left(1 - \frac{\varepsilon}{8}\right) t\|l\|^R \quad \text{for } t \geq t^* ,$$

where l' is the line parallel to l passing through the origin. Then by Proposition 3.1 the set of random variables $\xi_j = \tau^R(l', jt^*l) - \tau^R(l', (j+1)t^*l)$ satisfies the hypotheses of Lemma 2.2. Lemma 2.2 implies that

$$\mathbb{P}\left\{\tau^R(l', jt^*l) < \left(1 - \frac{\varepsilon}{6}\right) jt^*\|l\|^R\right\}$$

decays faster than any power of j. Due to periodicity of the flow, this implies (4.5) for any curve $\gamma \in \mathcal{C}_R$. □

PROOF OF THEOREM 1.1: By Theorem 3.6 and Lemma 4.10 for any bounded curve γ with positive diameter we almost surely have

$$(1-\varepsilon)t\mathcal{B} \subset \mathcal{W}_t(\gamma) \subset (1+\varepsilon)t\mathcal{B} \tag{4.6}$$

for all sufficiently large t.

The set $\mathcal{B}$ was defined to be a unit ball in the norm $\|\cdot\|^R$. However, $\mathcal{W}_t(\gamma)$ does not depend on R. Therefore, $\mathcal{B}$ does not depend on R. Inclusion (4.6) can be applied to a closed curve γ_1 containing a bounded set Ω inside, as well as to a continuous curve γ_2 contained inside Ω. Therefore, the statement of the theorem follows from inclusions (4.6). □

Appendix A: Control Theorem

PROOF OF THEOREM 2.3: We divide the proof into four steps:

- Step 1. Reduce the theorem for a general $\mathcal{X}$-simple control to a control with constant $v(t) \equiv 1$.
- Step 2. Further reduce it to a control consisting of just one vector field, e.g., X_1.
- Step 3. Change Wiener measure $\mathbb{P}$ on $(\theta_1(t), \dots, \theta_d(t)) \in \mathbb{R}^d$ to an equivalent one $\tilde{\mathbb{P}}$, which singles out the first component.
- Step 4. Show that under time rescaling $\tilde{\mathbb{P}}$ converges to a measure concentrated on the space of continuous path whose last $d-1$ components are identically zero.

A.1 Reduction to "Constant" Velocity

By definition it suffices to show that for any vector field $Y \in L_k$, any piecewise continuous function $v(t)$ on [0,1], and any $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that for $Z(t,x) = v(t)Y(x)$

$$\mathbb{P}\left\{\sup_{x\in\mathbb{T}^2,\, s\in[0,\delta_1]}\left|f_{0,s}x - \Phi\left(Z, \frac{s}{\delta_1}\right)x\right| < \varepsilon\right\} \geq \delta_2\,. \tag{A.1}$$

Since v can be approximated by piecewise constant functions, we see by rescaling that it is enough to establish (A.1) for $v \equiv 1$.

A.2 Reduction from a General Control $Z(t,x) = Y(x)$ to One Field Control $Z(x) = X_1(x)$

We prove (A.1) for the case $k = 2$. The general case can be proved using the same method. Let $Y \in L_2$. Renumerating $\{X_k\}$ if necessary, one can assume that $Y = X_1$, $Y = X_2$, or $Y = [X_1, X_2]$. In the latter case,

$$\Phi(Y,t) = \lim_{N\to\infty} \left\{\Phi\left(X_2, -\frac{t}{N}\right)\Phi\left(X_1, -\frac{t}{N}\right)\Phi\left(X_2, \frac{t}{N}\right)\Phi\left(X_1, \frac{t}{N}\right)\right\}^{N^2} .$$

Again rescaling the time, we see that it suffices to prove (A.1) in the case $v \equiv 1$ and $Y = X_1$ or X_2. It suffices to prove that for any $\varepsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$$\text{(A.2)} \qquad \mathbb{P}\left\{\sup_{x\in\mathbb{T}^2,\, s\in[0,\delta_1]} \left| f_{0,s}x - \Phi\left(X_1, \frac{s}{\delta_1}\right)x\right| < \varepsilon\right\} \geq \delta_2 .$$

A.3 Shift of the Wiener Measure

Let $w_1(t) = \theta_1(t) - t/\delta_1$. Then (1.1) becomes

$$dx_t = X_0(x_t)dt + \frac{1}{\delta_1}X_1(x_t)dt + X_1(x_t)\circ dw_1(t) + \sum_{k=2}^{d} X_k(x_t)\circ d\theta_k(t) .$$

Since $\theta_1 \to w_1$ is absolutely continuous in $C[0,\delta_1]$ with the Jacobian explicitly given by the Girsanov formula, to prove (A.2) it suffices to show that for any $\varepsilon > 0$ there exist $\delta_1, \delta_2' > 0$ such that

$$\text{(A.3)} \qquad \tilde{\mathbb{P}}\left\{\sup_{x\in\mathbb{T}^2,\, s\in[0,\delta_1]} \left| f_{0,s}x - \Phi\left(X_1, \frac{s}{\delta_1}\right)x\right| < \varepsilon\right\} \geq \delta_2' ,$$

where $\tilde{\mathbb{P}}$ is the Wiener measure on $(w_1, \theta_2, \dots, \theta_d)$. Rename w_1 to θ_1 again, and let $\mathcal{A}_1$ be the event that the solutions of

$$dx_t = \frac{1}{\delta_1}X_1(x_t)dt + X_0(x_t)dt + \sum_{k=1}^{d} X_k(x_t)\circ d\theta_k(t)$$

are ε-close to the solutions of $dx_t = (1/\delta_1)X_1(x_t)dt$ for $t \in [0,\delta_1]$. Thus we need to show that for every $\varepsilon > 0$ there exist $\delta_1, \delta_2' > 0$ such that

$$\text{(A.4)} \qquad \mathbb{P}\{\mathcal{A}_1\} \geq \delta_2.$$

A.4 Time Rescaling

After time change $t = \delta_1\tau$, we can rewrite (A.4) as

$$\text{(A.5)} \qquad \mathbb{P}\{\mathcal{A}_2\} \geq \delta_2',$$

where $\mathcal{A}_2$ is the event that the solutions of

$$\text{(A.6)} \qquad dx_\tau = X_1(x_\tau)d\tau + \delta_1 X_0(x_\tau)d\tau + \sqrt{\delta_1}\sum_{k=1}^{d} X_k(x_\tau)\circ d\theta_k(\tau)$$

are ε-close to the solutions of

$$dx_\tau = X_1(x_\tau)d\tau \tag{A.7}$$

for $t \in [0,1]$. However, as $\delta_1 \to 0$ the solutions of (A.6) converge weakly to the solutions of (A.7). So given positive ε and δ_2', we see that (A.5) holds if we choose δ_1 sufficiently small. □

Appendix B: Proof of Sweeping Lemma 3.7

PROOF: We divide the proof into six steps. Here is a brief outline of the proof.

- Step 1. Reduce the problem of sweeping the R-ball $B_R(A)$ to the problem of sweeping a little square $U \subset B_R(A)$.
- Step 2. Define a little square U.
- Step 3. Reduce the problem of sweeping a little square U with large probability to a problem of sweeping the little square U in a fixed time with positive probability. Proof of the latter step is decomposed into two stages.
- Step 4 (or Stage 1). Take a bigger square $C \supset U$ and using the strong Hörmander condition show that with positive probability the image of a long curve in a unit time connects ∂C with ∂U.
- Step 5 (or Stage 2). Using Theorem 2.3, reduce sweeping of a little box U to a control problem.
- Step 6. Construct a sweeping control.

B.1 From Sweeping the Ball $B_R(A)$ to Sweeping a Little Square $U = U(B)$

Since $B_R(A)$ is compact, it is enough to establish a local version of (3.9). In other words, it suffices to show that for any point $B \in B_R(A)$ there exists a neighborhood $U = U(B)$ such that if

$$\sigma_U = \inf\left\{t > 0 : U \subset \bigcup_{s\le t} \gamma_s\right\},$$

then for all m there is a constant C_m such that if γ satisfies the assumptions of Lemma 3.7, then

$$\mathbb{P}\{\sigma_U > t\} \le C_m t^{-m}\,. \tag{B.1}$$

B.2 Definition of a Little Square $U = U(B)$

Before giving the proof of (B.1), let us describe the choice of $U(B)$. By the strong Hörmander condition, given a point B there are vector fields $Y_1, Y_2 \in L$ that are transversal at B. We can choose coordinates $z = (z_1, z_2)$ near B so that B is at the origin,

$$Y_1 = \frac{\partial}{\partial z_1}\,, \quad Y_2 = a(z_1, z_2)\frac{\partial}{\partial z_2}\,, \quad a(0,0) = 1\,.$$

By shrinking the coordinate neighborhood if necessary, we can assume that $0.99 < a < 1.01$. By rescaling the coordinates, we can assume that

$$\text{Range}(z) = [-20, 20]^2 .$$

Let $U = z^{-1}([-1, 1]^2)$.

B.3 From Large to Positive Probability

Now we prove (B.1) for U defined in B.2. Select a point $A_0 \in \gamma$ such that $\text{dist}(A_0, A) \le R$. Let t_{j_n} be those of the stopping times t_j (defined in Section 3.1) that satisfy $\text{dist}(\gamma_{t_{j_n}}, A) \le R$. From Proposition 3.1 it follows that for any $m > 0$

$$\mathbb{E}(t_{j_n} - t_{j_{n-1}})^m < C_m \tag{B.2}$$

for some C_m. We claim that there exists $\theta > 0$ such that

$$\mathbb{P}\{\sigma_U < t_{j_n} \mid \sigma_U \ge t_{j_{n-1}}\} \ge \theta . \tag{B.3}$$

Formula (B.3) implies that

$$\mathbb{P}\{\sigma_U < t_{j_n}\} \ge 1 - \theta^n.$$

By Lemma 2.2 there exists a constant C such that for all m there is $\bar{C}_m$ with the property

$$\mathbb{P}\{t_{j_n} > Cn\} \le \bar{C}_m n^{-m} .$$

Since the last two inequalities imply (B.1), it remains to prove (B.3).

By definition $t_{j_n} - t_{j_{n-1}} \ge 1$. Hence (B.3) follows from the following estimate: there exists $\theta > 0$ such that for any long curve γ such that $\text{dist}(\gamma, A) \le R$, we have

$$\mathbb{P}\left\{U \subset \bigcup_{s \le 1} \gamma_s\right\} \ge \theta . \tag{B.4}$$

B.4 Stage 1 of Sweeping a Little Square U (Getting Close)

We shall now prove (B.4). Let

$$C(B) = z^{-1}([-5, 5]^2) .$$

Thus $U \subset C(B)$. Take two points $x', x'' \in \gamma$ such that $\text{dist}(x', B) \le R$, $\text{dist}(x', x'') = \frac{1}{2}$. By the strong Hörmander condition for the two-point motion there exists $p_0 > 0$ such that

$$\mathbb{P}\{ f_{0,1/2}x' \in U,\ f_{0,1/2}x'' \notin C(B)\} \ge p_0 .$$

Let $\hat{\gamma}$ be the piece of $\gamma_{1/2}$ joining $x'_{1/2}$ to $x''_{1/2}$. Let $\bar{\gamma}$ be a minimal subcurve of $\hat{\gamma}$ lying inside $C(B)$ and joining the boundary of $C(B)$ with $U(B)$ (minimality means that no proper subcurve of $\bar{\gamma}$ has these properties). By minimality $\bar{\gamma} \bigcap \partial C(B)$ is one point, which we call y.

$\partial C(B)$ consists of four segments corresponding to the four sides of the square. To fix our notation, assume that $y \in z^{-1}(\{-5\} \times [-5, 5])$. Other cases are similar. Let $\tilde{\gamma}$ be the minimal subcurve joining y to $z(\{-1\} \times [-5, 5])$ (see Figure B.1). In

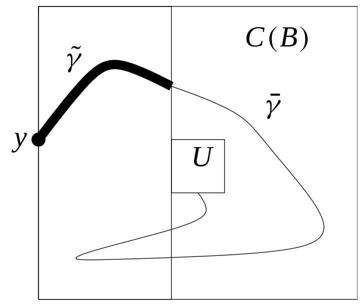

FIGURE B.1. Definition of $\tilde{\gamma}$.

order to prove (B.4), it suffices to prove the following: There exists $\theta > 0$ such that if $\tilde{\gamma}$ is any curve joining $z(\{-5\} \times [-5, 5])$ to $z(\{-1\} \times [-5, 5])$, then

$$\mathbb{P}\left\{U \subset \bigcup_{s=0}^{1/2} \tilde{\gamma}_s\right\} \geq \theta\,. \tag{B.5}$$

B.5 Stage 2: Sweeping a Little Square and Reduction to a Control Problem

Recall the definition of $\mathcal{X}$-simple control $Z(t, x)$ before Theorem 2.3. We shall construct a $\mathcal{X}$-simple control $Z(t, x)$ with the property that for any family $\{\Psi(s, x)\}_{s=0}^{1}$ of continuous maps of the plane such that

$$|z(\Psi(s, x)) - z(\Phi(Z, s)x)| < \frac{1}{2}\,, \quad \text{we have} \quad U \subset \bigcup_{x\in\tilde{\gamma}} \bigcup_{s=0}^{1} \Psi(s, x)\,. \tag{B.6}$$

Then Theorem 2.3 would imply (B.5).

Choose some parametrization $\tilde{\gamma} = \tilde{\gamma}(u)$, $u \in [0, 1]$. Let

$$\xi(s, u) = \Phi(Z, s)\tilde{\gamma}(u)\,, \quad \zeta(s, u) = \Psi(s, \tilde{\gamma}(u))\,.$$

We want to construct a control such that

$$U \subset \bigcup_{s=2/3}^{1} \bigcup_{u=0}^{1} \zeta(s, u) \tag{B.7}$$

for each Ψ satisfying (B.6). Let Γ denote the boundary of $[\frac{2}{3}, 1] \times [0, 1]$. To show (B.7) we exhibit a control such that for all $\tilde{B} \in U$ the index

$$\text{ind}(\zeta(\Gamma), \tilde{B}) = 1\,. \tag{B.8}$$

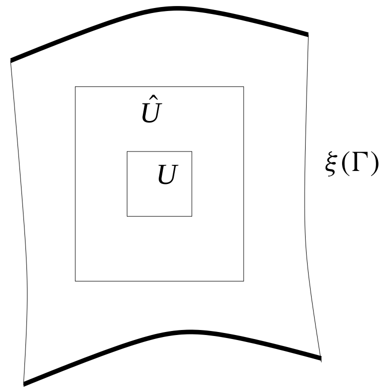

FIGURE B.2. Proof of Lemma 3.7. $\xi(\Gamma)$ consists of two almost translates of $\tilde{\gamma}$ and two almost vertical segments.

To obtain (B.8) for each ζ such that

$$|z(\zeta) - z(\xi)| < \frac{1}{2}, \tag{B.9}$$

we construct Z such that

$$\text{ind}(\xi(\Gamma), \tilde{B}) = 1 \quad \text{and} \quad \text{dist}(z(\xi(\Gamma)), z(U)) \geq 1 \tag{B.10}$$

(see Figure B.2). Note that (B.10) implies that any ζ satisfying (B.9) is homotopic to ξ in $\mathbb{R}^2 - \tilde{B}$ and so (B.8) holds.

B.6 Construction of a Sweeping Control

It remains to construct a control satisfying (B.10). Let $\hat{U} = z^{-1}[-2, 2]^2$. Let

$$Z(t, \cdot) = \begin{cases} -24Y_2(\,\cdot\,), & 0 \leq t < \frac{1}{3}, \\ 9Y_1(\,\cdot\,), & \frac{1}{3} \leq t < \frac{2}{3}, \\ 45Y_2(\,\cdot\,), & \frac{2}{3} \leq t \leq 1. \end{cases}$$

We claim that $Z(t, x)$ has the required properties. Let $\xi(s, u) = (a(s, u), b(s, u))$. Since $-5 \leq b(0, u) \leq 5$, it follows that $-5 - 8 \times 1.01 \leq b(2/3, u) \leq 5 - 8 \times 0.99$. The second inequality shows that $\xi(2/3, u)$ lies below $\hat{U}$. Similar computations show that $\xi(1, u)$ lies above $\hat{U}$, $\xi(s, 0)$ lies to the left of $\hat{U}$, and $\xi(s, 1)$ lies to the right of $\hat{U}$. This proves (B.10). □

Appendix C: Stable Manifolds and Cones

Here we prove Lemma 4.3 about local properties of stable manifolds. In this section $\mathbb{P}$ denotes the measure on solutions of (1.1) when x_0 is chosen according to the invariant measure on the torus and $\mathbb{P}_x$ denotes the measure on solutions of (1.1) where $x_0 = x$. We need to prove that for any $\varepsilon > 0$ there exist $a(\varepsilon)$, $b(\varepsilon)$, and $L(\varepsilon)$ such that

$$\mathbb{P}\{A_j(a, b, L)\} > 1 - \frac{\varepsilon}{10}, \quad j = 1, 2,$$

where A_j are defined by (4.2). We divide the proof into four steps.

- Step 1. Define geometric quantities of stable manifolds.
- Step 2. Establish probabilistic estimates for these geometric quantities.
- Step 3. State sufficient conditions for events in Lemma 4.3 to hold in terms of these quantities.
- Step 4. Reformulate these sufficient conditions and prove them.

C.1 Geometric Characteristics of Stable Manifolds

Given x, let

$$r(x, t) = \sup\left\{r > 0 : W^s(x, t) \cap \partial B_r(x) \neq \varnothing\right\}.^\dagger$$

Recall that $W_r^s(x, t)$ denotes a connected component of $W_r^s(x, t) \cap B_r(x)$. Given $r > 0$, let $\kappa(x, r, t)$ be the maximal curvature of $W_r^s(x, t)$ and

$$L(x, r, t) = \sup_{u \geq t} \operatorname{diam} f_{t,u} W_r^s(x, t).$$

By stationarity the distributions of $r(x, t)$, $\kappa(x, r, t)$, and $L(x, r, t)$ do not depend on t. We write $r(x)$, $\kappa(x, r)$, and $L(x, r)$ for $r(x, 0)$, $\kappa(x, r, 0)$, and $L(x, r, 0)$. By the stable manifold theorem, r is positive and both κ and L are finite almost surely. Therefore, for any r, $\varepsilon_0 > 0$ there exist positive r_0, κ_0, and L_0 such that

$$\mathbb{P}\{r(x) \geq r_0\} \geq 1 - \varepsilon_0,$$
$$\mathbb{P}\{\kappa(x, r) \leq \kappa_0\} \geq 1 - \varepsilon_0,$$
$$\mathbb{P}\{L(x, r) \leq L_0\} \geq 1 - \varepsilon_0.$$

In the next step of the proof, we use the strong Hörmander condition to conclude that these constants can be chosen independently of x.

C.2 Probabilistic Estimates on Geometric Characteristics of Stable Manifolds

LEMMA C.1 *For any r, $\varepsilon_0 > 0$ there exist positive r_1, κ_1, and L_1 such that for each point $x \in \mathbb{R}^2$ we have*

$$\mathbb{P}_x\{r(x) \geq r_1\} \geq 1 - \varepsilon_0,$$
$$\mathbb{P}_x\{\kappa(x, r) \leq \kappa_1\} \geq 1 - \varepsilon_0,$$
$$\mathbb{P}_x\{L(x, r) \leq L_1\} \geq 1 - \varepsilon_0.$$

† It can be shown that $r(x, t) = +\infty$ almost surely but we shall not use this fact.

PROOF: We prove the first statement only. Proofs of the other statements are similar.

By the strong Hörmander condition, which guarantees that the time one transition density for the process is a smooth positive function, and by the Markov property there is a C such that the following holds:

For any measurable set $\tilde{\Omega}$ on the space of the realizations of the flow with $\mathbb{P}\{\tilde{\Omega}\} > 1 - \varepsilon_0$ we have that $\mathbb{P}_x\{h_1\tilde{\Omega}\} \geq 1 - C\varepsilon_0$, where h_1 is the time one shift on the space of the realizations of the flow.

Let $\tilde{r}$ be such that

$$\mathbb{P}\{r(y,1) \geq \tilde{r}\} \geq 1 - \frac{\varepsilon_0}{2C}$$

(here y is considered to be uniformly distributed on the torus). Then for any x we have

$$\mathbb{P}_x\{r(x_1,1) \geq \tilde{r}\} \geq 1 - \frac{\varepsilon_0}{2}\,.$$

Since $f_{0,1}$ is a diffeomorphism, there exists N such that

$$\mathbb{P}\{\| f_{0,1}\|_{C^1} \leq N\} \geq 1 - \frac{\varepsilon_0}{2}\,. \tag{C.1}$$

However, if

$$\| f_{0,1}\|_{C^1} \leq N\,,$$

then $f_{0,1}^{-1}$ cannot decrease lengths by more than a factor of N (since $f_{0,1}$ can increase lengths by at most a factor of N.) Hence our claim follows with $r_1 = \tilde{r}/N$. □

Lemma C.1 implies the second part of (4.2). Now we proceed to establish the first part of (4.2).

C.3 Sufficient Geometric Conditions for the First Event of (4.2)

Recall that $K^x(a,b,n)$ is the cone appearing in the first event of (4.2). We claim that for any r, κ and $\alpha < \frac{\pi}{2}$ there exist a and b such that if

$$\angle(W^s(x), n) \leq \alpha\,, \tag{C.2a}$$

$$\kappa(x,r) \leq \kappa\,, \tag{C.2b}$$

$$r(x) \geq r\,, \tag{C.2c}$$

then the first event of (4.2) holds.

To establish the claim, consider coordinate system z_1, z_2 such that x is at the origin and n coincides with the z_1-axis. Let v_1, v_2 be the coordinates of the unit tangent vector to $W^s(x)$ pointing inside K^x and let σ be the arc length parametrization of $W^s(x)$. Then by (C.2a)

$$v_1 \geq \cos(\alpha)\,, \quad |v_2| \leq \sin(\alpha)\,,$$

and so by (C.2b)

$$\frac{dz_1}{d\sigma} \geq \cos(\alpha) - \kappa\sigma\,, \quad \left|\frac{dz_2}{d\sigma}\right| \leq \sin(\alpha) + \kappa\sigma\,.$$

Hence

$$z_1 \geq \cos(\alpha)\sigma - \frac{\kappa\sigma^2}{2}\,, \quad |z_2| \leq \sin(\alpha)\sigma + \frac{\kappa\sigma^2}{2}\,.$$

Therefore we can choose a, b so that z_1 reaches a before $z_1/\sqrt{z_1^2+z_2^2}$ reaches b. This implies that the half of $W_a^s(x)$ lying inside K^x crosses K_3^x but avoids K_1^x and K_2^x. Similarly, the second part of $W_a^s(x)$ lies in $-K^x$ and so never crosses K^x. This proves the claim.

Note that Lemma C.1 implies that (C.2b) and (C.2c) hold with probability arbitrarily close to 1 for appropriate r and κ. It remains to show that by increasing α we can make the probability of (C.2a) arbitrarily close to 1.

C.4 Sufficient Condition for (C.2a) and Proof

Let $E_s(x)$ denote the stable direction at x. In view of the above claim, in order to establish (4.2), it remains to show that for any ε_0 there is $\alpha < \pi/2$ such that for any x we have

$$\mathbb{P}\{\angle(E_s(x), n) \leq \alpha\} \geq 1 - \varepsilon_0\,. \tag{C.3}$$

More generally, we shall show that there exists a constant $\bar{C}$ such that for all (x, v) and all ε there exists β such that

$$\mathbb{P}\{\angle(E_s(x), v) < \beta\} \leq \varepsilon_0\,. \tag{C.4}$$

Applying (C.4) with v orthogonal to n we obtain (C.3).

To establish (C.4), observe that by the strong Hörmander condition there exists a unique invariant measure $\tilde{\mu}$ on the unit tangent bundle $S\mathbb{T}^2$, and this measure has a smooth density. Let

$$\mathcal{B}(\beta, \varepsilon) = \{(x, v) : \mathbb{P}\{\angle(E_s(x), v) < \beta\} > \varepsilon\}\,.$$

Note that for fixed x the cardinality of the largest 2β-separated set inside $\{v : (x, v) \in \mathcal{B}(\beta, \varepsilon)\}$ is less than $1/\varepsilon$. Thus

$$\tilde{\mu}(\mathcal{B}(\beta, \varepsilon)) \leq \text{const}\,\frac{\beta}{\varepsilon}\,.$$

Thus

$$\mathbb{P}\{(x_1, df_{0,1}v) \in \mathcal{B}(\beta_1, \varepsilon)\} \to 0 \quad \text{as } \beta_1 \to 0\,.$$

Hence we can find β_1 such that

$$\mathbb{P}\left\{\angle(TW^s(x_1, 1), df_{0,1}v) < \beta_1\right\} < \frac{\varepsilon_0}{2}\,.$$

To conclude, we need the following elementary fact: Given $Q \in \mathrm{SL}_2(\mathbb{R})$, consider its action on the projective line $Q(v) = Qv/\|Qv\|$. Then the derivative of this action is given by $DQ(v)v' = P_{Qv}(Qv')/\|Qv\|$, where P_{Qv} is the orthogonal projection in the direction of Qv. In particular,

$$\|DQ\| < \|Q\|^2\,. \tag{C.5}$$

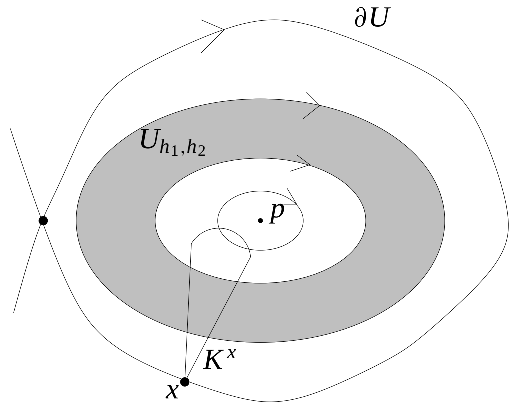

FIGURE D.1

By (C.5) there exists an absolute constant C such that if

$$\angle\big(TW^s(x_1,1), df_{0,1}v\big) \ge \beta_1 \quad \text{then} \quad \angle(E_s(x), v) \ge \frac{\beta_1}{C\|df_{0,1}^{-1}\|^2}\,.$$

Take N such that

$$\mathbb{P}\{\|df^{-1}\| \ge N\} \le \frac{\varepsilon_0}{2}\,.$$

Then (C.4) follows with $\beta = \beta_1/(CN^2)$. This completes the proof.

Appendix D: The Cone K^x and the Images of the Curves γ_1 and γ_2

In this appendix we prove Lemma 4.4. Recall the notation used in the proof of Proposition 4.2 (see Figure D.1).

As in the proof of Lemma 3.7, it is sufficient to prove the following statement: there are $\theta > 0$ and $T > 0$ such that

$$\mathbb{P}\big\{\overline{\gamma_i} \cap K_1^x \ne \varnothing,\ \overline{\gamma_i} \cap K_2^x \ne \varnothing,\ i = 1, 2\big\} \ge \theta$$

for any $\gamma_1, \gamma_2 \in \mathcal{C}_{2R}$ for some $t \le T$. Without loss of generality, we may assume that $H_1(x) = 0$ when $x \in \partial U$ and that $H_1(x) > 0$ for $x \in U$. Let $U_{h_1,h_2} = \{x : x \in U,\ h_1 \le H_1(x) \le h_2\}$. Note that for h_1 and h_2 small enough, U_{h_1,h_2} is homeomorphic to an annulus. Since the time it takes a solution of $dy_t/dt = X_1(y_t)$, $y_0 = x$, to make one rotation along the stream line tends to infinity when $\mathrm{dist}(x, \partial U) \to 0$, for sufficiently small h_1 and h_2 we can introduce the angle-action coordinates in U_{h_1,h_2} such that the dynamics under the flow X_1 is described by

$$\dot\phi = \omega(I)\,, \quad \dot I = 0\,, \quad \phi \in [0, 2\pi]\,, \quad I \in [h_1, h_2]\,,$$

with the property that

$$\omega(h_2) > \omega(h_1)\,. \tag{D.1}$$

We say that a curve fully crosses U_{h_1,h_2} if it is contained in the closure of U_{h_1,h_2} and its endpoints belong to $\{H_1 = h_1\}$ and $\{H_1 = h_2\}$, respectively. For a curve γ in U_{h_1,h_2}, we define its winding number $w(\gamma)$ as the change of the ϕ-coordinate over the curve. In this notation $|w(K_1^x)|, |w(K_2^x)| \le 1$. Thus, if $|w(\gamma)| \ge 2$, then γ crosses both K_1^x and K_2^x. Also, for any $v \in \mathbb{R}$ and any curve γ fully crossing U_{h_1,h_2}, we have

$$w(\Phi(vX_1, 1)\gamma) - w(\gamma) = v(\omega(h_2) - \omega(h_1)) ,$$

where $\Phi(vX_1, t)$ is the flow generated by $\dot{x} = vX_1(x)$.

Therefore, due to (D.1), there exists $v > 0$ such that for any γ fully crossing U_{h_1,h_2}, at least two of the three curves

$$\Phi(vX_1, 1)\gamma , \quad \Phi(2vX_1, 1)\gamma , \quad \Phi(3vX_1, 1)\gamma \tag{D.2}$$

have winding numbers larger than 2 in absolute value, and therefore cross both K_1^x and K_2^x. The same is true for any curves sufficiently close to those in (D.2).

Therefore, by Theorem 2.3 we conclude that with positive probability for every pair of curves $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$, both of which fully cross U_{h_1,h_2}, the time δ_1 images of the curves under the action of the stochastic flow (1.1) cross both K_1^x and K_2^x.

It remains to prove the following:

LEMMA D.1 *There exist positive c_1 and T such that for all $\gamma_1, \gamma_2 \in \mathcal{C}_{2R}$ we have*

$$\mathbb{P}\{\mathcal{A}\} \ge c_1 ,$$

where $\mathcal{A}$ is the event that for some $t < T$ both $f_{0,t}\gamma_1$ and $f_{0,t}\gamma_2$ contain subcurves that fully cross U_{h_1,h_2}.

PROOF: Similarly to the way it was done in the proof of Proposition 3.1 for one curve, it is easy to show that there is a sequence of stopping times τ_j such that $\tau_{j+1} - \tau_j \ge 1$, $\mathbb{E}(\tau_{j+1} - \tau_j)^m \le C_m$, and each of the curves $\gamma_i(\tau_j)$ is long and intersects $B_R(0)$. Take $A_{ij} \in \gamma_i(\tau_j)$ such that

$$\operatorname{dist}\left(A_{1j} + \mathbb{Z}^2, A_{2j} + \mathbb{Z}^2\right) \ge \frac{1}{10} .$$

Let U_b and U_{inf} be the bounded and the unbounded components of $\mathbb{R}^2 - U_{h_1,h_2}$, respectively. By the strong Hörmander condition

$$\mathbb{P}\{ f_{\tau_j,\tau_j+1/2}A_{ij} \in U_b,\ i = 1, 2\} \ge c_2 . \tag{D.3}$$

By Corollary 2.6 we can choose $N \in \mathbb{N}$ such that

$$\mathbb{P}\{\operatorname{diam}(\gamma_i(t)) > \operatorname{diam}(U) \text{ for all } t > N,\ i = 1, 2\} \ge 1 - \frac{c_2}{3} . \tag{D.4}$$

Then (D.3) and (D.4) imply that

$$\mathbb{P}\{\gamma_i(\tau_{N+1/2}) \text{ fully cross } U_{h_1,h_2}\} \ge \frac{c_2}{3} .$$

Choose T such that

$$\mathbb{P}\left\{\tau_N + \frac{1}{2} \leq T\right\} \leq \frac{c_2}{6}.$$

Then Lemma D.1 follows with $c_1 = c_2/6$. □

Appendix E: Morse Functions on the Two-Dimensional Torus

In this appendix we present some basic facts about so-called Morse functions on the two-dimensional torus $\mathbb{T}^2$. An excellent account about Morse functions is Milnor's book [13]. Let $H : \mathbb{T}^2 \to \mathbb{R}$ be a C^∞ smooth function on $\mathbb{T}^2$, let $C^\infty(\mathbb{T}^2)$ be the space of such functions with C^∞ topology, and $x = (x_1, x_2) \in \mathbb{T}^2$ be a standard coordinate system. A point $x \in \mathbb{T}^2$ is called *critical* if the gradient of H vanishes at x, i.e., $\nabla H(x) = (\partial_{x_1} H(x), \partial_{x_2} H(x)) = 0$. A function H is called a *Morse* function if all its critical points are nondegenerate, i.e., the Hessian matrix $\partial^2_{x_i x_j} H(x)$ has full rank. It follows from the definition that critical points are isolated. Each critical point is either a local minimum (respectively, maximum) or a saddle point. In particular, we have that each Morse function has only finitely many critical points.

LEMMA E.1 *There is an open and dense set of Morse functions in $C^\infty(\mathbb{T}^2)$. In other words, a generic function is Morse.*

PROOF: As we shall see, this lemma follows by the transversality theorem. To each smooth map $H : \mathbb{T}^2 \to \mathbb{R}$ one associates a so-called 2-jet $j^2 H = (H, \nabla H, \partial^2_{x_i x_j} H) : \mathbb{T}^2 \to \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^4\}$. Now consider condition $\Sigma = \{x : \nabla H(x) = 0$ and rank $\partial^2_{x_i x_j} H(x) < 2\}$. This condition consists of three independent equations or, equivalently, it has codimension 3, which is greater than the dimension of $\mathbb{T}^2$. By the transversality theorem [1], the property that the image of $j^2 H$ does not intersect Σ is open and dense. But if $j^2 H$ misses Σ, then H is Morse. □

A different way to prove this lemma is in [13, sec. 5].

An image of a critical point under H is called a *critical value*. All the other values in the image $H(\mathbb{T}^2)$ are called *regular*. It follows from the theorem on implicit functions that the preimage (or level set) $L_a = H^{-1}(a) \subset \mathbb{T}^2$ of regular value a is a smooth curve. Since each Morse function has only finitely many critical points, it has only finitely many critical values.

LEMMA E.2 *Let a be a regular value of a C^∞ smooth Morse function. Then for small $\varepsilon > 0$ and any $|a' - a| < \varepsilon$, level sets $L_{a'} = H^{-1}(a')$ are smooth curves. Moreover, each connected component of $H^{-1}([a-\varepsilon, a+\varepsilon])$ is diffeomorphic to a cylinder $(\phi, I) \in [0,1] \times \mathbb{T}$ with level sets $L_{a'}$ being circles $\{I = I_{a'}\}$.*

PROOF: Since critical points are isolated, for each regular value a all nearby values a' are regular. By the implicit function theorem for these a' level sets, $\{L'_a\}_{a'}$ are smooth curves and depend smoothly on a'. Therefore, we could choose local coordinates in $H^{-1}([a-\varepsilon, a+\varepsilon])$ so that one coordinate parametrizes the value of H and the other parametrizes the length of the corresponding level curves. □

Notice that if X is a vector field on $\mathbb{T}^2$ and H is its stream function, then trajectories of X belong to level sets of H. Therefore, if in this lemma we choose time parametrization on level curves (circles), then we get *action-angle coordinates*. Namely, the vector field X in the new coordinate system becomes $\dot{\phi} = \omega(I),\ \dot{I} = 0$ for some smooth function $\omega(I)$.

LEMMA E.3 *Let U be the maximal open set containing $H^{-1}([a-\varepsilon, a+\varepsilon]) \subset U \subset \mathbb{T}^2$ where action-angle coordinates can be defined. Then the boundary ∂U contains a saddle.*

PROOF: By maximality ∂U must contain critical points. If all those points were maxima and minima, then the closure $\bar{U}$ would be diffeomorphic to the 2-sphere, a contradiction. The result follows. □

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Bibliography

[1] Arnol′d, V. I.; Guseĭn-Zade, S. M.; Varchenko, A. N. *Singularities of differentiable maps*. Vol. I. The classification of critical points, caustics and wave fronts. Monographs in Mathematics, 82. Birkhäuser, Boston, 1985.

[2] Baxendale, P. H. Lyapunov exponents and relative entropy for a stochastic flow of diffeomorphisms. *Probab. Theory Related Fields* **81** (1989), no. 4, 521–554.

[3] Baxendale, P. H.; Stroock, D. W. Large deviations and stochastic flows of diffeomorphisms. *Probab. Theory Related Fields* **80** (1988), no. 2, 169–215.

[4] Bony, J.-M. Principe du maximum, inégalite de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés. *Ann. Inst. Fourier (Grenoble)* **19** (1969), fasc. 1, 277–304.

[5] Carverhill, A. Flows of stochastic dynamical systems: ergodic theory. *Stochastics* **14** (1985), no. 4, 273–317.

[6] Cranston, M.; Scheutzow, M.; Steinsaltz, D. Linear expansion of isotropic Brownian flows. *Electron. Comm. Probab*. **4** (1999), 91–101.

[7] Cranston, M.; Scheutzow, M.; Steinsaltz, D. Linear bounds for stochastic dispersion. *Ann. Probab*. **28** (2000), no. 4, 1852–1869.

[8] Dolgopyat, D.; Kaloshin, V.; Koralov, L. Sample path properties of the stochastic flows. *Ann. Probab*. **32** (2004), no. 1A, 1–27.

[9] Dolgopyat, D.; Kaloshin, V.; Koralov, L. Hausdorff dimension in stochastic dispersion. *J. Statist. Phys*. **108** (2002), no. 5-6, 943–971.

[10] Léandre, R. Minoration en temps petit de la densité d'une diffusion dégénérée. *J. Funct. Anal*. **74** (1987), no. 2, 399–414.

[11] Lisei, H.; Scheutzow, M. Linear bounds and Gaussian tails in a stochastic dispersion model. *Stoch. Dyn.* **1** (2001), no. 3, 389–403.

[12] Mattila, P. *Geometry of sets and measures in Euclidean spaces*. Fractals and rectifiability. Cambridge Studies in Advanced Mathematics, 44. Cambridge University Press, Cambridge, 1995.

[13] Milnor, J. *Morse theory*. Annals of Mathematics Studies, 51. Princeton University Press, Princeton, N.J., 1963.

[14] Scheutzow, M.; Steinsaltz, D. Chasing balls through martingale fields. *Ann. Probab.* **30** (2002), no. 4, 2046–2080.

[15] Stroock, D. W.; Varadhan, S. R. S. On the support of diffusion processes with applications to the strong maximum principle. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971)*, vol. 3: Probability theory, 333–359. University of California Press, Berkeley, Calif., 1972.

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