FORMULAS FOR MATH475 FINAL.

Enumeration.

(1) Permutations $P_{n,r} = \frac{n!}{(n-r)!}$

(2) Combinations $C(n, r) = \left(\begin{array}{c} n \\ r \end{array} \right) = \frac{n!}{r!(n-r)!}$

(3) Distributions of n objects into r groups with repetition: n^r

(4) Distributions of n objects into r groups without repetition: $C(n + r - 1, r)$

(5) Distributions of n objects into r groups with restricted repetition: $P(n, r_1, r_2 \ldots r_m) = \frac{n!}{r_1!r_2!\ldots r_m!}$.

Common generating functions

(1) $\frac{1-x^m}{1-x} = 1 + x + \ldots x^{m-1}$.

(2) $\frac{1}{1-x} = 1 + x + \ldots x^n + \ldots$.

(3) $(1 \pm x)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array} \right) x^r (\pm 1)^r$.

(4) $\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} \left(\begin{array}{c} n + r - 1 \\ r \end{array} \right) x^r$.

Operations with generating functions.

If $a_n \to f(x)$, $b_n \to g(x)$ then

(1) $a_n + b_n \to f(x) + g(x)$,

(2) $\sum_{r=0}^{n} a_r b_{n-r} \to f(x)g(x)$.

(3) $\sum_{r=0}^{n} a_n \to \frac{f(x)}{1-x}$.

(4) $a_{n-k} \to \frac{f(x) - \sum_{r=0}^{k-1} a_r x^r}{x^k}$.

Solving recurrence relations.

(1) $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_r a_{n-r} \Rightarrow a_n = \sum \lambda_j^n P_j(n)$ where λ_j are roots of

$$\lambda^r = c_1 \lambda^{r-1} + c_2 \lambda^{r-2} + \cdots + c_r$$

and $P_j(n)$ are polynomial of degree $m_j - 1$ where m_j is the multiplicity of λ_j.

(2) $a_n = ca_{n-1} + f(n) \Rightarrow a_n = c^n (a_0 + \sum_{j=1}^{n} c^{-j} f(j))$.

(3) $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots c_r a_{n-r} + f(n) \Rightarrow a_n = a_n^* + \tilde{a}_n$ where a_n^* is the general solution of the homogeneous recurrence (1) and $\tilde{a}_n$ is a particular solution.

(4) $f(n) = \lambda^n P(n) \Rightarrow \tilde{a}_n = \lambda^n Q(n)$ where

$$\text{degree}(Q) = \text{degree}(P) + m$$

where m is the multiplicity of λ.

Inclusion-Exclusion formula. $S_r := \sum_{i_1, i_2 \ldots i_r} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}|$.

(1) $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{r=1}^{n} (-1)^{r-1} S_r$.

(2) Exactly m occurrences: $N_m = \sum_{r=m}^{n} (-1)^{r-m} \binom{r}{m} S_r$.

(3) At least m occurrences: $N_m^* = \sum_{r=m}^{n} (-1)^{r-m} \binom{r-1}{m-1} S_r$.