MATH475 SAMPLE EXAMS.

Exam 1.

(1) How many ways are there to distribute 8 different toys and 8 identical candy to 3 children
(a) without restrictions;
(b) if first child should get exactly 2 toys;
(c) if the first child should get at least 2 toys?

Solution. (a) There are 3^8 ways to distribute toys and $\binom{8+3-1}{3-1} = \binom{10}{2}$ ways to distribute candy (using the formulas for ordered and unordered selection with repetition respectively). Therefore the answer is $3^8 \times \binom{10}{2}$.

(b) Here the number of ways to distribute toys changes. Namely there are $\binom{8}{2}$ ways to select the toys for the first child and 2^6 ways to distribute remaining toys among the two other children. Therefore the answer is $\binom{8}{2} \times 2^6 \times \binom{10}{2}$.

(c) The complimentary event is that the first child gets either 0 or 1 toy. Arguing as in part (b) we conclude that there are

$$3^8 - \binom{8}{1} \times 2^7 - \binom{8}{0} \times 2^8$$

ways to distribute toys. Therefore the answer is

$$\left[3^8 - 8 \times 2^7 - 2^8\right] \times \binom{10}{2}.$$

(2) (a) Write a generating function for the number of ways a given number n can be represented as a sum of two odd and three even numbers.
(b) Find a number of ways 50 can be represented as a sum of two odd and three even numbers.

Solution. (a) We have

$$f(x) = (x + x^3 + x^5 + \dots + x^{2k+1} + \dots)^2 (1 + x^2 + x^4 + \dots + x^{2k} + \dots)^3.$$

Summing geometric progressions we find

$$x+x^3+x^5+\cdots+x^{2k+1}+\cdots=\frac{x}{1-x^2}\quad 1+x^2+x^4+\cdots+x^{2k}+\cdots=\frac{1}{1-x^2}.$$

Therefore

$$f(x)=\frac{x^2}{(1-x^2)^5}.$$

(b) Substituting $y=x^2$ into the formula

$$\frac{1}{(1-y)^5}=\sum_{k=0}^{\infty}\binom{k+4}{4}y^k$$

we get

$$\frac{x^2}{(1-x^2)^5}=x^2\sum_{k=0}^{\infty}\binom{k+4}{4}x^{2k}=\sum_{k=0}^{\infty}\binom{k+4}{4}x^{2k+2}.$$

Coefficient in front of x^{50} corresponds to $2k+2=50$, that is $k=24$. Thus the answer is $\binom{24+4}{4}=\binom{28}{4}$.

(3) Find a bounded solution to the recurrence relation

$$2a_n=5a_{n-1}-2a_{n-2},\quad a_0=1.$$

The characteristic polynomial is $2\lambda^2-5\lambda+2$. The roots are 2 and 1/2. Therefore the general solution takes form $a_n=A2^n+B\left(\frac{1}{2}\right)^n$. The solution is bounded if and only if $A=0$. From the the initial condition we have $a_0=B=1$. Therefore $a_n=\left(\frac{1}{2}\right)^n$.

(4) A bridge hand is called weak in a suit if it has neither ace nor king nor queen in this suit.

(a) Find a number of bridge hands which are weak in at least one suit.

(b) Find a number of bridge hands which are weak in exactly one suit.

Solution. Let A_1 be all hands which are weak in hearts, A_2 be all hands which are weak in diamonds, A_3 be all hands which are weak in spades and A_4 be all hands which are weak in clubs.

Then $A_1 = \binom{49}{13}$ since there are three forbidden cards (ace, king and queen of hearts). Likewise $A_1A_2 = \binom{46}{13}$, $A_1A_2A_3 = \binom{43}{13}$, $A_1A_2A_3A_4 = \binom{40}{13}$.

Therefore $S_1 = \binom{4}{1}\binom{49}{13}$, $S_2 = \binom{4}{2}\binom{46}{13}$, $S_3 = \binom{4}{3}\binom{43}{13}$, $S_4 = \binom{4}{4}\binom{40}{13}$.

Hence the answer to (a) is

$$\binom{4}{1}\binom{49}{13} - \binom{4}{2}\binom{46}{13} + \binom{4}{3}\binom{43}{13} - \binom{4}{4}\binom{40}{13}$$

and the answer to (b) is

$$\binom{4}{1}\binom{49}{13} - 2\binom{4}{2}\binom{46}{13} + 3\binom{4}{3}\binom{43}{13} - 4\binom{4}{4}\binom{40}{13}.$$

Exam 2.

(1) How many arrangements of the letters PEPPERMILL
(a) do not start and end with the same letter;
(b) have at least one P in the same position as in the original word;
(c) have all Es in the first half?

Solution. (a) There are $P_{10;3,2,2,1,1,1}$ arrangements of the letters PEPPERMILL. Among those $P_{8;1,2,2,1,1,1}$ start and end with P (since only places from 2d to 9th need to be specified), $P_{8,3,2,1,1,1}$ start and end with E and $P_{8,3,2,1,1,1}$ start and end with L. Therefore the answer is

$$\frac{10!}{3!2!2!} - \frac{8!}{2!2!} - 2\frac{8!}{3!2!}.$$

(b) Let A_1 be all arrangements with P at the first place, A_3 be all arrangements with P at the third place, A_4 be all arrangements with P at the fourth place. Arguing as in part (a) we get $|A_1| = \frac{9!}{2!2!2!}$, $|A_1 A_2| = \frac{8!}{2!2!}$ $|A_1 A_2 A_3| = \frac{7!}{2!2!}$. By inclusion exclusion formula

$$|A_1 \bigcup A_2 \bigcup A_3| = \binom{3}{1} \frac{9!}{2!2!2!} - \binom{3}{2} \frac{8!}{2!2!} + + \binom{3}{3} \frac{7!}{2!2!}.$$

(c) There are $\binom{5}{2}$ ways to choose places for E and $P_{8;3,2,1,1,1}$ to arrange remaining letters. Therefore the answer is

$$\binom{5}{2} \frac{8!}{3!2!}.$$

(2) Find the generating function for a sequence satisfying

$$a_n = 7a_{n-1} - 5a_{n-4} + n, \quad a_0 = 0, \quad a_1 = 1, \quad a_2 = 2, \quad a_3 = 3.$$

Solution. We have

$$\sum_{n=4}^{\infty} a_n x^n = \sum_{n=4}^{\infty} 7a_{n-1} x^n - \sum_{n=4}^{\infty} 5a_{n-4} x^n + \sum_{n=4} n x^n.$$

The left hand side equals

$$f(x) - \sum_{n=0}^{3} a_n x^n = f(x) - (x + 2x^2 + 3x^3).$$

On the right hand side the first term is $x(f(x) - (x + 2x^2)$ the second term is $x^4 f(x)$ and the last term is

$$x \sum_{n=4}^{\infty} n x^{n-1} = x \left(\sum_{n=4}^{\infty} x^n \right)' = x \left(\frac{x^4}{1-x} \right)' = \frac{4x^4 - 3x^5}{(1-x)^2}.$$

Solving the equation

$$f(x) - (x + 2x^2 + 3x^3) = x[f(x) - (x + 2x^2)] + x^4 f(x) + \frac{4x^4 - 3x^5}{(1-x)^2}$$

we get

$$f(x) = \frac{x - x^2 + 3x^4 - 2x^5}{(1-x)^2(1-x-x^4)}.$$

(3) Consider the following algorithm for finding two largest among n numbers. Divide the numbers into two equal groups, find the largest in the first $(M_1 > m_1)$ and the second $(M_2 > m_2)$ group and then find the two largest numbers among M_1, M_2, m_1, m_2 by using at most three comparisons (m_1 to M_2, m_2 to M_1 and M_1 to M_2). Find the recurrence relation for the number of comparisons needed in the worst case and solve it for $n = 2^k$.

Solution. The recurrence relation is

$$a_n = 2a_{n/2} + 3, \quad a_2 = 1.$$

Letting $b_k = a_{2^k}$ we get

$$b_k = 2b_{k-1} + 3, \quad b_1 = 1.$$

A constant C satisfies this relation if $C = 2C + 3$, that is $C = -3$. Thus the general solution of our relation is

$$b_k = A2^k - 3.$$

From $1 = b_1 = 2A - 3$ we get $A = 2$. Therefore $b_k = 2 \times 2^k - 3$ Since $n = 2^k$ we have $a_n = 2n - 3$.

(4) How many ways are there to put 10 identical red balls and 10 different blue balls into 4 boxes so that there is a box containing
(a) exactly 2 red balls;
(b) exactly 2 blue balls?

Solution. (a) Let A_1 be all distributions of red balls with exactly 2 balls in the first box, A_2 be all distributions with exactly 2 balls in the second box, A_3 be all distributions with exactly 2 balls in the third box, A_4 be all distributions with exactly 2 balls in the fourth box.

If there are exactly two balls in the first box then we need to distribute 8 remaining balls into three boxes so

$$|A_1| = \binom{8+3-1}{3-1} = \binom{10}{2} = 45.$$

Likewise

$$|A_1A_2| = \binom{7}{1} = 7,$$

$$|A_1A_2A_3| = 1, \quad |A_1A_2A_3A_4 = 0.$$

Therefore by inclusion exclusion formula

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = \binom{4}{1} \times 45 - \binom{4}{2} \times 7 + \binom{4}{3} = 142.$$

For each such arrangement of red ball there are 4^{10} arrangements of the blue balls. Therefore the answer is 142×4^{10}.

(b) We proceed as in part (a) interchanging the roles of blue and red balls. If there are exactly two blue balls in the first box then we need to decide which balls to put in the first box ($P_{10;2,8}$ possibilities) and then distribute remaining 8 balls into remaining three boxes (3^8 possibilities). Thus

$$|A_1| = P_{10;2,8} \times 3^8.$$

Likewise

$$|A_1A_2| = P_{10;2,2,6} \times 2^6, \quad |A_1A_2A_3| = P_{10;2,2,2,4} \times 1^4, \quad |A_1A_2A_3A_4| = 0.$$

Therefore by inclusion exclusion formula there are

$$\binom{4}{1}\frac{10!}{2!8!}3^8 - \binom{4}{2}\frac{10!}{2!2!6!}2^6 + \binom{4}{3}\frac{10!}{2!2!2!4!}1^6$$

ways to distribute blue balls. For each distribution there are

$$\binom{10+4-1}{4-1} = \binom{13}{3}$$

ways to distribute red balls. Therefore the answer is

$$\left[\binom{4}{1}\frac{10!}{2!8!}3^8 - \binom{4}{2}\frac{10!}{2!2!6!}2^6 + \binom{4}{3}\frac{10!}{2!2!2!4!}1^6\right]\binom{13}{3}.$$