ON SMALL GAPS IN THE LENGTH SPECTRUM

DMITRY DOLGOPYAT AND DMITRY JAKOBSON

Abstract. We discuss upper and lower bounds for the size of gaps in the length spectrum of negatively curved manifolds. For manifolds with algebraic generators for the fundamental group, we establish the existence of exponential lower bounds for the gaps. On the other hand, we show that the existence of arbitrary small gaps is topologically generic: this is established both for surfaces of constant negative curvature (Theorem 3.1), and for the space of negatively curved metrics (Theorem 4.1). While arbitrary small gaps are topologically generic, it is plausible that the gaps are not too small for almost every metric. One result in this direction is presented in Section 5.

1. Introduction: geodesic length separation in negative curvature

On negatively curved manifolds, the number of closed geodesics of length $\leq T$ grows exponentially in T. (We refer the reader to [Mar04, P-P, P-S] for a comprehensive discussion about the growth and distribution of closed geodesics).

The abundance of closed geodesics leads to the natural question about the sizes of gaps in the length spectrum. In the current note we present a number of results related to this question. In some situations we are able to control the gaps from below, while in other we show that such control is not possible in general.

We note that a presence of exponentially large multiplicities in the length spectrum of a Riemannian manifold (which can be considered as a limiting case of small gaps) changes the level spacings distribution of Laplace eigenvalues on that manifold, see e.g. [L-S].

For generic Riemannian metrics, the length spectrum is *simple* [Abr, A2], so for any closed geodesic γ, only γ^{-1} will have the same length. So, by the Dirichlet box principle, there exist *exponentially small* gaps between the lengths of different geodesics.

Accordingly, it seems interesting to investigate manifolds where the gaps between the lengths of different geodesics have exponential *lower bound:* there exist constants $C, \beta > 0$, such that for any $l_1 \neq l_2 \in \mathrm{Lsp}(M)$ (length spectrum of the negatively curved manifold M), we have

$$|l_2 - l_1| > Ce^{-\beta \cdot \max(l_1, l_2)}. \tag{1.1}$$

This assumption is satisfied for arithmetic hyperbolic groups by the trace separation criterion (cf. [Tak] and [Hej, §18]). In Section 2 we explain (see Theorem 2.6) why the assumption (1.1) holds for hyperbolic manifolds whose fundamental group has algebraic elements.

In particular, the surfaces satisfying (1.1) form a dense set in the corresponding Teichmuller space. On the other hand the existence of arbitrary small gaps is *topologically generic* as is shown in Theorem 3.1 for surfaces of constant negative curvature and in Theorem 4.1 for the space of negatively curved metrics endowed with C^r-topology, for any $r > 0$.

While arbitrary small gaps are topologically generic, it is plausible that the gaps are not too small for almost every metric. One result in this direction is presented in Section 5 there we obtain an explicit lower bound for the gaps valid for almost every hyperbolic surface.

Length separation between closed geodesics is relevant for the study of wave trace formulas on negatively-curved manifolds: to accurately study contributions from exponentially many closed

2000 *Mathematics Subject Classification.* Primary 37C25, 53C22; Secondary 20H10, 37C20, 37D20, 53D25.

D. D. was supported by NSF. D. J. was partially supported by NSERC, FQRNT and Peter Redpath fellowship.

geodesics to the wave trace formula, it is necessary to separate contributions from geodesics which differ either on the length axis, or in phase space. We remark that a suitable version of (1.1) always holds in *phase space*: small tubular neighbourhoods of closed geodesics in phase space are disjoint, as shown in [JPT]. Since there exist metrics for which the size of the length gaps cannot be controlled (Theorem 4.1), the authors in [JPT] established microlocal wave trace formula, and used the separation of closed trajectories in phase space in the proof.

2. Diophantine results for hyperbolic manifolds.

2.1. **Distances between algebraic numbers.** In this section we consider gaps in the length spectrum for manifolds whose fundamental group admits algebraic generators. But first we provide a few general results about the algebraic numbers.

Lemma 2.1. *If α is a root of $P(x) = x^D + a_{D-1}x^{D-1} + \cdots + a_0$ then*

$$|\alpha| \geq \frac{|a_0|}{\left(1 + \sum_{j=0}^{D-1} |a_j|\right)^{D-1}}.$$

Proof. Let α_j be the roots of P counted with multiplicities. We claim that $|\alpha_j| \leq R := 1 + \sum_j |a_j|$. Indeed if $|x| > R$ then since $R > 1$ we get

$$|P(x)| \geq |x|^D - \sum_{j=0}^{D-1} |a_j||x|^j \geq |x|^{D-1}\left(|x| - \sum_{j=0}^{D-1} |a_j|\right) > 0.$$

The result follows since $\prod_j |\alpha_j| = |a_0|$. □

Given a field K which is an extension of $\mathbf{Q}$ of degree d let $\mathfrak{H}(L, N, p)$ be the set of all elements of K of the form $\frac{\beta}{N^p}$ where $\beta \in \mathbf{O}_K$ and for each automorphism σ_j of K we have $|\sigma_j(\beta)| \leq L$.

Lemma 2.2. *If $0 \neq \alpha \in \mathfrak{H}(L, N, p)$ then*

$$|\alpha| \geq \frac{1}{L^{d-1}N^p}.$$

Proof. Indeed $|\beta|L^d \geq 1$ because $\prod_{j=1}^{d} |\sigma_j(\beta)| \geq 1$. □

Let $\mathfrak{I}(L, N, p, D)$ be the set of numbers which satisfy

$$\alpha^E + a_{E-1}\alpha^{E-1} + \cdots + a_0 = 0$$

where $E \leq D$ and $a_j \in \mathfrak{H}(L, N, p)$.

Corollary 2.3. *If $0 \neq \alpha \in \mathfrak{I}(L, N, p, D)$ then*

$$|\alpha| \geq \frac{1}{L^{d-1}N^p(DL+1)^{D-1}}.$$

Proof. Since $\alpha \neq 0$ we can assume after possibly reducing the degree of the polynomial that $a_0 \neq 0$. Then the result follows by combining Lemmas 2.1 and 2.2 □

Proposition 2.4. *(see e.g.* [vW, Section 17.2]*) There exists constants C and q such that if $\alpha_1, \alpha_2 \in \mathfrak{I}(L, N, p, D)$ then $\alpha_1 + \alpha_2$ and $\alpha_1 - \alpha_2$ are in $\mathfrak{I}(CL^q, N, pq, D^2)$.*

Combining Proposition 2.4 with Corollary 2.3 we obtain

Corollary 2.5. *If $\alpha_1, \alpha_2 \in \mathfrak{I}(L, N, p, d)$ then either $\alpha_1 = \alpha_2$ or*

$$|\alpha_1 - \alpha_2| \geq \frac{c}{L^{q(d-1)}N^{pq}L^{D^2}}.$$

2.2. **Manifolds with algebraic generators of π_1.** We now formulate the main result of this section.

Theorem 2.6. *Let X be a hyperbolic manifold such that the generators of $\pi_1(X)$ belong to $PSO_{n,1}(\bar{\mathbf{Q}})$. Then* (1.1) *holds.*

We remark that in dimension 2 groups satisfying the assumptions of Theorem 2.6 form a dense set in the corresponding Teichmuller space T_g. This can be established, for example, by the arguments of Section 5.

If $n \geq 3$ then [G-R, Theorem 0.11] building on earlier results of of Selberg [Sel] and Mostow ([Most]) shows the conditions of Theorem 2.6 are satisfied for all finite volume hyperbolic manifolds. Hence we obtain

Corollary 2.7. (1.1) *holds for finite volume hyperbolic manifolds of dimension $n \geq 3$.*

The proof of Theorem 2.6 is similar to the proof of Proposition 3 in [GJS], where it is shown that the rotation matrices in $\mathrm{SU}(2) \cap M_2(\overline{\mathbf{Q}})$ satisfy the *Diophantine condition* defined in [GJS]. Related results for other Lie groups were established in [ABRS, Br11, Var]. Related questions were also discussed in [Glu].

2.3. **Proof of Theorem 2.6.**

Proof. Let γ_1 and γ_2 be two closed geodesics. Let l_j be the length of γ_j, W_j be the word fixing γ_j, B_j be the matrix corresponding to W_j, m_j be the word length of W_j and $r_j = l_j/2$. To establish (1.1) it suffices to show that

$$|e^{r_1} - e^{r_2}| \geq \bar{C} e^{-\bar{c}\max(r_1, r_2)}. \tag{2.8}$$

Without a loss of generality we assume that $m_j \gg 1$. By ([Miln, Lemma 2]) we know that

$$\frac{l_j}{C} \leq m_j \leq C l_j \tag{2.9}$$

so (1.1) if trivial unless m_j and m_2 are comparable. Let us assume to fix our ideas that $m_2 \geq m_1$. By assumption there is a finite extension K of $\mathbf{Q}$ and numbers L and N such that all entries of the generators belong to $\mathfrak{H}(L, N, 1)$. Accordingly the entries of B_j belong to

$$\mathfrak{H}((L(n+1))^{m_j}, N, m_j).$$

Closed geodesics on X correspond to *loxodromic* elements of $\pi_1(X) \subset PSO_{n,1}$ (also called *boosts*) that that fix no points in $\mathcal{H}^n$ and fix two points in $\partial\mathcal{H}^n$. It is shown in the proof of [F-LJ, Thm. I.5.1] that B_j has precisely two positive real eigenvalues $\alpha_{1,j} = e^{r_j}$ and $\alpha_{2,j} = e^{-r_j}$; all other eigenvalues of B_j have modulus one. Since the coefficients of the characteristic polynomial of B_j are the sums of minors we have

$$e^{r_j} \in \mathfrak{I}((L(n+1))^{(n+1)m_j}(n+1)!, N, m_j(n+1)).$$

Reducing to the common denominator we see that both e^{r_1} and e^{r_2} belong to

$$\mathfrak{I}((L(n+1))^{(n+1)m_2} N^{m_2 - m_1}(n+1)!, N, m_2(n+1)).$$

Now (2.8) follows by Corollary 2.5 and (2.9). □

Remark 2.10. In dimension two the proof can be simplified slightly by remarking that $2\cosh(l_j/2) = \mathrm{tr} B_j \in K$. An alternative proof of Theorem 2.6 could proceed by using explicit formulas for the lengths of closed geodesics on hyperbolic manifolds (see e.g. [P-R, (3), p. 246]) and the estimates for linear forms in logarithms (see e.g. [B-W, Chapter 2]). The proof we give is more elementary, using only basic facts about algebraic numbers and matrix eigenvalues; and fairly concrete.

3. Small gaps for surface of constant negative curvature.

Let

$$G_g = \{(A_1, \dots A_{2g}) \in (SL_2(\mathbf{R}))^{2g} : [A_1, A_2][A_3, A_4] \dots [A_{2g-1}, A_{2g}] = I\}.$$

Theorem 3.1. *The set of tuples $(A_1, A_2 \dots A_{2g}) \in G_g$ where (1.1) fails is topologically generic.*

Proof. Let γ_A denote the closed geodesic whose lift to the fundamental cover joins q and Aq. Let $\mathbb{L}$ denote the length spectrum of the geodesics γ_A where A belongs to a subgroup generated by A_1 and A_2. Note that for a dense set of tuples it holds that for each δ there exists L such that for $l > L$ the set $[l, l + \delta]$ intersects $\mathbb{L}$. One way to see this is to consider the geodesics $\gamma_{A_1^k A_2^m}$. Their length have asymptotics

$$\kappa(A_1, A_2)k\lambda_1 + m\lambda_2$$

where e^{λ_j} is the leading eigenvalue of A_j. Note that for typical A_1, A_2 we have $\kappa(A_1, A_2) \neq 0$ and λ_1 and λ_2 are non commensurable. Consider a geodesic $\bar\gamma = \gamma_{A_3 A_1^n}$ where n is very large. By the foregoing discussion there exists $l \in \mathbb{L}$ such that $|l - L_{\bar\gamma}| < \delta$. Now consider the perturbations of A_3 of the form $A_3(\eta) = \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} A_3$. Assume that $A_3 A_1^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. After applying a small perturbation if necessary we can assume that all entries of this matrix have the same order as its trace. Then

$$\operatorname{tr}(A_3(\eta)A_1^n) = \operatorname{tr}(A_3 A_1^n) + \eta c,$$

so by a small perturbation we can make $L_{\gamma_{A_3(\eta)A_1^n}}$ as close to l as we wish. Now the result follows by a standard Baire category argument (cf. Section 4). □

4. Constructing metrics with small gaps in the length spectrum

This section is devoted to the proof of the following fact.

Theorem 4.1. *For any $r > 3$ for any negatively-curved C^r metric g, for any function $F(t)$ (which we assume is monotone and fast decreasing), and a number $\delta > 0$, there exists a metric $\bar g$, such that $||\bar g - g||_{C^r} < \delta$ and there exists an infinite sequence of pairs of closed $\tilde g$-geodesics $\gamma_{1,j}, \gamma_{2,j}$ with $L_{\bar g}(\gamma_{i,j}) \to \infty$ as $j \to \infty$, and*

$$|L_{\bar g}(\gamma_{1,j}) - L_{\bar g}(\gamma_{2,j})| < \min\{F(L_{\bar g}(\gamma_{1,j})), F(L_{\bar g}(\gamma_{2,j}))\}. \tag{4.2}$$

This shows that, in general, one *cannot* obtain good lower bounds for gaps in the length spectrum for a C^r open set of negatively curved metrics.

Theorem 4.1 follows from the lemma below by a standard Baire category argument.

Lemma 4.3. *Given a metric g and numbers L and δ there is a metric $\tilde g$ such that $||g - \tilde g||_{C^r} \leq \delta$ and there are two $\tilde g$-geodesics γ_1 and γ_2 such that*

$$L_{\tilde g}(\gamma_1) = L_{\tilde g}(\gamma_2) > L.$$

We also need the following fact

Lemma 4.4. *Let g and $\tilde g$ be two negatively curved metrics such that $||g - \tilde g||_\infty \leq \delta$ and γ and $\tilde\gamma$ be two closed geodesics for g and $\tilde g$ respectively of lengths L and $\widetilde L$. If γ and $\tilde\gamma$ are homotopic then*

$$\frac{L}{1+\delta} \leq \widetilde L \leq L(1+\delta).$$

Proof. Recall that for negatively curved there exists a unique geodesic in each homotopy class and this geodesic is length minimizing. The second inequality follows since the length of γ with respect to $\tilde g$ is at most $L(1 + \delta)$ and $\tilde\gamma$ is shorter. The first inequality follows from the second by interchanging the roles of g and $\tilde g$. □

Proof of Theorem 4.1. We claim that given metric g and numbers $k \in \mathbb{N}$ and $\delta > 0$ there exists a metric $\bar{g}$ such that $||\bar{g} - g||_{C^r} < \delta$ and for each $j = 1 \dots k$ there are geodesics $\gamma_{1,j}, \gamma_{2,j}$ such that

$$L_{\bar{g}}(\gamma_{i,j}) > j, \quad |L_{\bar{g}}(\gamma_{1,j}) - L_{\bar{g}}(\gamma_{2,j})| \leq F(\max(L_{\bar{g}}(\gamma_{1,j}), L_{\bar{g}}(\gamma_{2,j}))). \tag{4.5}$$

It follows that the space of metrics satisfying (4.2) is topologically generic and hence dense.

It remains to construct $\bar{g}$ satisfying (4.5).

By Lemma 4.3 we can find g_1 such that $||g - g_1||_{C^r} < \frac{\delta}{2}$ and there are two geodesics $\gamma_{1,1}$ and $\gamma_{2,1}$ such that

$$L_{g_1}(\gamma_{i,1}) > 1 \text{ and } L_{g_1}(\gamma_{1,j}) = L_{g_1}(\gamma_{2,j}).$$

For $j \geq 1$ we apply Lemma 4.3 to find g_j such that

$$||g_j - g_{j-1}||_{C^r} \leq \min\left(\frac{\delta}{2^j}, \min_{l=1}^{j-1} \frac{F(L_{g_l}(\gamma_{1,l})) + 1)}{L_{g_l}(\gamma_{1,l}) 2^{j-l+1}}\right)$$

and there are two geodesics $\gamma_{1,j}, \gamma_{2,j}$ such that

$$L_{g_j}(\gamma_{1,j}) = L_{g_j}(\gamma_{2,j}) > j.$$

Then g_k satisfies the required properties since, by Lemma 4.4, the lengths of $g_{i,l}$ have changed by less than $F(L_{g_l}(\gamma_{1,l}) + 1)/2$ in the process of making consecutive inductive steps. □

Remark 4.6. In particular if we continue the above procedure for the infinite number of steps then the limiting metric will satisfy the conditions of Theorem 4.1.

The proof of Lemma 4.3 relies on two facts. If γ is a closed geodesic let ν_γ denote the invariant measure for the geodesic flow supported on γ. Let h denote the topological entropy of the geodesic flow. Let μ denote the Bowen-Margulis measure. Recall [P-P] that μ the measure of maximal entropy for the geodesic flow. It has a full support in the unit tangent bundle SM.

Lemma 4.7. [P-P, Theorem 6.9 and Proposition 7.2] *$Lhe^{-Lh} \sum_{L(\gamma) \leq L} \nu_\gamma$ converges as $L \to \infty$ to μ.*

Lemma 4.8. *For each $q_0 \in M$ there exists ε such that for each L there is periodic geodesic γ such that $L(\gamma) > L$ and γ does not visit an ε neighborhood of q_0.*

Proof of Lemma 4.3. Pick a small $\bar{\delta}$ and large L. By Lemma 4.8 there exists a closed geodesic γ_1 such that $L_g(\gamma_1) > L$ and $d(q(\gamma_1(t)), q_0) > \varepsilon)$. Let γ_2 be a closed geodesic such that $L_g(\gamma_1) < L_g(\gamma_2) < L_g(\gamma_1) + \bar{\delta}$ and γ spends at least time $\mu(B(q_0, \varepsilon/2)/2L_g(\gamma_1)$ inside $B(q_0, \varepsilon/2)$ (the existence of such a geodesic follows from Lemma 4.7). Take $\tilde{g}^\eta = (1 - \eta z(q))g$ where $z(q) = 1$ on $B(q_0, \varepsilon/2)$ and $z(q) = 0$ outside $B(q_0, \varepsilon)$. We can choose z so that $||z||_{C^r} = O(\varepsilon^{-r})$. Then $||g - \tilde{g}^\eta||_{C^r} = O(\eta/\varepsilon^r)$. Let γ_2^η be the closed geodesic for $\tilde{g}^\eta$ homotopic to γ_2. Note that γ_1 is a geodesic of $\tilde{g}^\eta$ for each η and $L_{\tilde{g}^\eta}(\gamma_1) \equiv L_g(\gamma_1)$. Also

$$L_{\tilde{g}^\eta}(\gamma_2^\eta) \leq L_{\tilde{g}^\eta}(\gamma_2) \leq L_g(\gamma_1) + \bar{\delta} - \frac{\mu(B(q_0, \varepsilon/2) L_g(\gamma_1)\eta}{2}.$$

Accordingly there exists $\eta < \frac{2\bar{\delta}}{L\mu(B(q_0, \varepsilon/2))}$ such that $L_{\tilde{g}^\eta}(\gamma_1^\eta) = L_{\tilde{g}^\eta}(\gamma_2)$ as claimed. □

In the proof of Lemma 4.8 we need several facts about the dynamics of the geodesic flow which we call ϕ_t. Recall [A1] that ϕ_t is uniformly hyperbolic. In particular, there is a cone field $\mathcal{K}(x)$ and $\lambda > 0$ such that for $u \in \mathcal{K}$, $||d\phi_t(u)|| \geq e^{\lambda t}||u||$. Moreover the cone field $\mathcal{K}$ can be chosen in such a way that if $x = (q, v)$ and $u = (\delta q, \delta v) \in \mathcal{K}(x)$ then

$$||\delta q|| \geq c||\delta v|| \text{ and } \angle(\delta q, v) \geq \frac{\pi}{4} \tag{4.9}$$

We call a curve σ *unstable* if $\dot{\sigma} \in \mathcal{K}$. By the foregoing discussion if σ is un unstable curve then the length of the projection of $\phi_t(\sigma)$ on M is longer than $ce^{\lambda t}$.

Proof of Lemma 4.8. We first show how to construct a not necessary closed geodesic avoiding $B(q_0, \varepsilon)$ and then upgrade the result to get the existence of a closed geodesics.

The first part of the argument is similar to [B-S, D]. Pick a small $\kappa > 0$. Take an unstable curve σ of small length κ. We show that if κ and ε are sufficiently small then σ contains a point such that the corresponding geodesic avoids $B(q_0, \varepsilon)$. Let T_1 be a number such that $|\phi_{T_1}(\sigma)| = 1$ where ϕ denotes the geodesic flow. Note that $T_1 = O(|\ln \kappa|)$. Also observe that due to (4.9) there exists a number r_0 such that if $\tilde{\sigma}$ is an unstable curve and $x \in \tilde{\sigma}$ is such that $d(q(x), q_0) < \varepsilon$ then for all $y \in \tilde{\sigma}$ such that $C\varepsilon \le \mathbf{d}(y, x) \le r_0$ we have

$$\mathbf{d}(q(\phi_t y), q_0) > \varepsilon$$

for $|t| < r_0$ where $\mathbf{d}$ denotes the distance in the phase space (just take r_0 much smaller than the injectivity radius of q_0).

Thus the set

$$\{y \in \phi_{T_1}(\sigma) : d(q(\phi_{-t} y), q_0) \le \varepsilon \text{ for some } 0 \le t \le T_1\}$$

is a union of $O(|\ln \kappa|/r_0)$ components of length $O(\varepsilon/\kappa^a)$ for some $a > 0$. Therefore if $\kappa \ll 1$ and $\varepsilon \ll \kappa$ then the average distance between the components is much larger than κ. So we can find $\sigma_1 \subset \phi_{T_1}\sigma$ such that $|\sigma_1| = \kappa$, and if $y \in \sigma_1$ then $d(q(\phi_{-t} y), q_0) > \varepsilon$ for each $0 \le t \le T_1$. Take T_2 such that $|\phi_{T_2}\sigma_1| = 1$. Then we can find $\sigma_2 \subset \phi_{T_2}\sigma_1$ such that $|\sigma_2| = \kappa$, and if $y \in \sigma_2$ then $d(q(\phi_{-t} y), q_0) > \varepsilon$ for each $0 \le t \le T_2$. We continue this procedure inductively to construct arcs σ_j for all $j \in \mathbb{N}$. Taking

$$x = \bigcap_{j=1}^{\infty} \phi_{-(T_1+T_2+\cdots+T_j)} \ \ \sigma_j$$

we obtain a geodesic avoiding $B(q_0, \varepsilon)$. To complete the proof we need

Lemma 4.10. *(Anosov Closing Lemma) (see* [H-K, Section 18]*) Given $\eta > 0$ there exists $\delta > 0$ such that if for some t_1, t_2 such that $|t_2 - t_1|$ is sufficiently large we have $\mathbf{d}(\gamma(t_1), \gamma(t_2) < \delta$ then there exists a closed geodesic $\tilde{\gamma}$ such that $|L(\tilde{\gamma}) - |t_2 - t_1|| < \eta$ and for each $t \in [t_1, t_2]$ there exists s such that $\mathbf{d}(\gamma(t), \tilde{\gamma}(s)) < \eta$.*

Take δ corresponding to $\eta = \varepsilon/2$. Consider points $\gamma(jL)$ where $j = 1 \dots K$. By pigeonhole principle if K is sufficiently large we can find j_1, j_2 such that $\mathbf{d}(\gamma(j_1 L), \gamma(j_2 L)) < \delta$ and so there exists a closed geodesic $\tilde{\gamma}$ avoiding $B(q_0, \varepsilon/2)$. Since ε is arbitrary, Lemma 4.8 follows. □

Suppose now that $\dim(M) = 2$. Let $\mathcal{H}_r(M)$ denote the space of C^r metrics with positive topological entropy. This set is C^r open ([K]) and dense. (If $\text{genus}(M) \ge 2$ then every metric has positive topological entropy [K]. For torus the density of $\mathcal{H}_r(M)$ follows from [Ban] and for sphere it follows from [K-W]).

Theorem 4.11. *The set of metrics satisfying* (4.2) *is topologically generic in $\mathcal{H}(M)$.*

Corollary 4.12. *The set of metrics satisfying* (4.2) *is topologically generic in the space of all metrics on M.*

Proof of Theorem 4.11. By [K] if $g \in \mathcal{H}_r(M)$ then there is a hyperbolic basic set Λ for the geodesic flow. Since Lemmas 4.3, 4.7, 4.8 and 4.10 remain valid in the setting of hyperbolic sets the proof is similar to the proof of Theorem 4.2. (In the proof of Lemma 4.8 we need to take σ_1 so that it crosses completely an element of some Markov partition Π such that all elements of Π have unstable length between κ and $C\kappa$. The number of eligible segments now is not $O(1/\kappa)$ but $O(1/\kappa^a)$ for some $a > 0$ but this is still much larger than $|\ln \kappa|$.) □

5. Small gaps for hyperbolic surfaces, continued

Here we show that for Lebesgue-typical hyperbolic surface the gaps in the length spectrum cannot be too small. Our argument in similar to [K-R]. Related results are obtained in [Var].

5.1. **Small values of polynomials.**

Proposition 5.1. *(see e.g* [M-H, Section 3.2]*) Consider a degree D polynomial $P(x) = a_D x^D + a_{D-1}x^{D-1} + \dots a_0$. Then*

$$\sup_{[-1,1]} |P(x)| \geq \frac{|a_D|}{2^{D-1}}.$$

Corollary 5.2. *Let $0 \neq P \in \mathbf{Z}[x_1, x_2 \dots x_n]$, $\deg(P) = D$ then*

$$\sup_{[-1,1]^n} |P(x)| \geq \frac{1}{2^{D-1}}.$$

Proof. By induction. For $n = 0$ or 1 the result follows from Proposition 5.1.

Next, suppose the statement is proven for polynomials of $n-1$ variables. If P does not depend on x_n then we are done. Otherwise let $k > 0$ be the degree of P with respect to x_n. Then

$$P(x) = a_k(x_1, \dots, x_{n-1})x_n^k + a_{k-1}(x_1, \dots, x_{n-1})x_n^{k-1} + \dots + a_0(x_1, \dots, x_{n-1})$$

where a_k is the polynomial with integer coefficients of degree $D-k$. Let

$$(\bar{x}_1, \dots \bar{x}_{n-1}) = \arg \max_{[-1,1]^{n-1}} |a_k(x_1, \dots, x_{n-1})|.$$

Then

$$\sup_{[-1,1]^n} |P(x_1, \dots x_{n-1}, x_n)| \geq \max_{x_n \in [-1,1]} |P(\bar{x}_1, \dots \bar{x}_{n-1}, x_n)| \geq |a(\bar{x}_1, \dots, \bar{x}_{n-1})| 2^{1-k}$$

$$\geq 2^{1+k-D} 2^{1-k} = 2^{2-D}$$

completing the proof. □

Proposition 5.3. (Remez inequality) *(see* [B-G] *or* [Yom, Theorem 1.1]*) Let B be a convex set in $\mathbf{R}^n$, $\Omega \subset B$, and P be a polynomial of degree D. Then*

$$\sup_B |P| \leq C_B \mathrm{mes}^{-D}(\Omega) \sup_\Omega |P|.$$

Corollary 5.4. *Under the conditions of Proposition 5.3*

$$\mathrm{mes}(x \in B : |P(x)| \leq \varepsilon) \leq \left(\frac{C_B \varepsilon}{\sup_B |P|} \right)^{1/D}.$$

Proof. Apply Proposition 5.3 with $\Omega = \{x \in B : |P(x)| \leq \varepsilon\}$. □

Corollary 5.5. *If $P_N \in \mathbf{Z}[x_1, x_2, \dots, x_n]$ are polynomials of degree D_N and ε_N is a sequnces such that $\sum_N \varepsilon^{1/D_N} < \infty$ then $|P(x_1, \dots x_n)| < \varepsilon_N$ has only finitely many solutions for almost every $(x_1 \dots x_n) \in \mathbf{R}^n$.*

Proof. It suffices to show this for a fixed cube B with side 2. Then Corollaries 5.2 and 5.4 give

$$\mathrm{mes}(x \in B : |P_N(x)| \leq \varepsilon_N) \leq \left(C 2^{D_N} \varepsilon_N\right)^{1/D_N} = \bar{C} \varepsilon_N^{1/D_N}$$

so the statement follows from Borel-Cantelli Lemma. □

5.2. **Polynomial maps on $SL_2(\mathbf{R})$.**

Corollary 5.6. *Let m be a fixed number.*

(a) Let $P_N \in \mathbf{Z}((a_1,b_1,c_1,d_1),\dots,(a_m,b_m,c_m,d_m))$ be polynomials of degree D_N. For $A_1,\dots A_m \in SL_2(\mathbf{R})$ with $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ let

$$H_N(A_1,\dots,A_m) = P_N((a_1,b_1,c_1,d_1),\dots,(a_m,b_m,c_m,d_m)).$$

If $\sum_N \varepsilon_N^{1/((m+2)D_N)} < \infty$ then $|H_N(A_1\dots A_m)| < \varepsilon_N$ for only finitely many N for almost every $(A_1,\dots A_m) \in (SL_2(\mathbf{R}))^m$.

(b) Given $g \in \mathbb{N}$ let

$$G_g = \{(A_1,\dots A_{2g}) \in (SL_2(\mathbf{R}))^{2g} : [A_1,A_2][A_3,A_4]\dots[A_{2g-1},A_{2g}] = I\}.$$

Let $P_N \in \mathbf{Z}((a_1,b_1,c_1,d_1),\dots,(a_{2m},b_{2m},c_{2m},d_{2m}))$ be polynomials of degree D_N. Let

$$H_N(A_1,\dots,A_{2g}) = P_N((a_1,b_1,c_1,d_1),\dots,(a_{2g},b_{2g},c_{2g},d_{2g})).$$

Assume that H_N is not identically equal to 0 on G_g. If $\sum_N \varepsilon_N^{\delta_N} < \infty$ where $\delta_N = \frac{1}{(4g-2)(g+2)D_N}$ then $|H_N(A_1\dots A_{2g})| < \varepsilon_N$ for only finitely many N for almost every $(A_1,\dots A_{2g}) \in G_g$.

Proof. (a) It suffices to prove the statement under the assumption that $|a_j| > \delta$ for some fixed $\delta > 0$. Then $d_j = \frac{1+b_jc_j}{a_j}$ and so

$$H(A_1,\dots A_m) = \frac{\widetilde{P}_N((a_1,b_1,c_1),\dots(a_m,b_m,c_m))}{\prod_{j=1}^m a_j^{d_N}}$$

where $\widetilde{P}_N$ is a polynomial of degree $\widetilde{d_n} \le (m+2)d_N$. Thus if $|P_N| \le \varepsilon_N$ then $|\widetilde{P}_N| \le \widetilde{\varepsilon}_N := \frac{\varepsilon_N}{\delta^{(m+2)D_N}}$. Since

$$\sum_N \widetilde{\varepsilon}_N^{1/\widetilde{d}_N} \le \frac{1}{\delta}\sum_N \varepsilon_N^{1/(m+2)D_N} < \infty$$

the result follows from Corollary 5.5.

(b) Rewriting the equations defining G_g in the form

$$[A_1,A_2]\dots[A_{2g-3},A_{2g-2}]A_{2g-1}A_{2g}A_{2g-1}^{-1} = A_{2g}$$

we can express the entries of A_{2g} as rational functions of the entries of the other matrices. Arguing as in part (a) we can reduce the inequality $|P_N(A_1,\dots A_{2g-1},A_{2g})| < \varepsilon$ to $|\widehat{P}_N(A_1,\dots A_{2g-1})| < \widehat{\varepsilon}_N$ where $\widehat{P}_N$ is the polynomial of degree $(4g-2)D_N$. Now the result follows from part (a). □

Corollary 5.7. *For each $\eta > 0$ for almost every $A_1,\dots A_m \in SL_2(\mathbf{R})$ the inequality*

$$||W(A_1,\dots A_m) - I|| > (2m-1)^{-|W|^2(m+2+\eta)}$$

holds for all except for finitely many words W.

Proof. If $||W(A_1,\dots A_m) - I|| \le \varepsilon$ then all entries of $W - I$ are ε close to I. Conisdering for example, the condition $W_{11}(A_1,\dots A_m) - 1$ we get a polynomial of degree $|W|$. Therefore, by Corollary 5.6 it suffices to check that

$$\sum_W (2m-1)^{-\frac{|W|^2(m+2+\eta)}{|W|(m+2)}} < \infty$$

but the above sum equals to

$$\sum_D (2m-1)^D (2m-1)^{-D-D(\eta/(m+2)} = \sum_D (2m-1)^{-\eta D/(m+2)} < \infty. \square$$

Corollary 5.8. *For $\mathcal{A} = (A_1 \dots A_{2g}) \in G_g$ let $S_{\mathcal{A}}$ be the surface defined by $\mathcal{A}$. Given a word W let $l(W, \mathcal{A})$ be the length of the closed geodesic in the homotopy class defined by W. Then for each $\eta > 0$ the following holds for almost all $\mathcal{A} \in G_g$*

There exists a constant $K = K(\mathcal{A})$ such that for each pair W_1, W_2 either

$$l(W_1, \mathcal{A}) = l(W_2, \mathcal{A}) \text{ or}$$

$$|l(W_1, \mathcal{A}) - l(W_2, \mathcal{A})| \geq K(4g-1)^{-[(2g+4)(4g-2)+\eta]\max^2(|W_1|,|W_2|)}. \tag{5.9}$$

Remark 5.10. Recall that [Ran] shows that for any hyperbolic surface the length spectrum has unbounded multiplicity so there are many pairs of non conjugated words there the first alternative of the corollary holds.

Remark 5.11. Note that (2.9) shows that $l(W_1, \mathcal{A})$ can be close to $l(W_2, \mathcal{A})$ only if the lengths of W_1 and W_2 are of the same order. Thus (5.9) implies that for almost every $\mathcal{A}$ there are constants K, R such that

$$|l(W_1, \mathcal{A}) - l(W_2, \mathcal{A})| \geq Ke^{-R(\mathcal{A})l^2(W_1,\mathcal{A})}.$$

Proof. Let $P_W(\mathcal{A}) = \operatorname{tr}(W(\mathcal{A}))$. Since $P_W(\mathcal{A}) = 2\cosh(l(W, \mathcal{A})/2)$, it follows that if $l(W_1(\mathcal{A}))$ is close to $l(W_2(\mathcal{A}))$ then

$$l(W_1, \mathcal{A}) - l(W_2, \mathcal{A})| \geq C|P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})|e^{-|W_1|D}.$$

Therefore it suffices to show that if $l(W_1, \mathcal{A}) \neq l(W_2, \mathcal{A})|$ then

$$|P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})| \geq \widetilde{K}e^{-|W_1|D}(4g-1)^{-[(2g+4)(4g-2)+\eta]\max^2(|W_1|,|W_2|)}.$$

Since η is arbitrary, we can actually check that

$$|P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})| \geq K(4g-1)^{-[(2g+4)(4g-2)+\eta]\max^2(|W_1|,|W_2|)}.$$

To verify this we will show that for almost all $\mathcal{A} \in G_m$ the inequality

$$|P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})| < (4g-1)^{-[(2g+4)(4g-2)+\eta]\max^2(|W_1|,|W_2|)}$$

has only finitely many solutions. Let $P_{W_1,W_2}(\mathcal{A}) = P_{W_1}(\mathcal{A}) - P_{W_2}(\mathcal{A})$. It is a polynomail of degree $\max(|W_1|, |W_2|)$. So by Corollary 5.6(b) it suffices to check that

$$\sum_{W_1,W_2} (4g-1)^{-\frac{[(2g+4)(4g-2)+\eta]\max(|W_1|,|W_2|)}{(4g-2)(g+2)}} < \infty$$

There are at most $(4g-1)^{2k}$ pairs (W_1, W_2) with $k = \max(W_1, W_2)$ so the last sum is estimated by

$$\sum_k (4g-1)^{2k}(4g-1)^{-\frac{[(2g+4)(4g-2)+\eta]k}{(4g-2)(g+2)}} = \sum_k (4g-1)^{-\frac{\eta k}{(4g-2)(g+2)}} < \infty$$

proving the result. □

6. Open problems.

(1) A suitable version of Theorem 2.6 should hold for other symmetric spaces. In particular, recall that arithmetic manifolds appear as fundamental domains G/Γ where G is a connected semi-simple algebraic $\mathbf{R}$-group without compact factors of $\mathbf{R}$-rank ≥ 2, and Γ is a lattice in G (cf. [Mar74, Mar75, Mar77]). Thus we expect that a version of Theorem 2.6 should hold in higher rank setting. Note however, that for higher rank symmetric spaces closed orbits are not isolated but appear in families.

(2) The proof of Theorem 4.1 relies on localized perturbations. Therefore it does not work in the analytic category. We expect that Theorem 4.1 is still valid for analytic metrics but the proof would require new ideas.

(3) It is likely that an explicit lower bound for the gaps in the length spectrum could also be obtained for *prevalent* set of negatively curved metrics (see [Kal] for related results) but we do not pursue this question here.

Acknowledgements.

The authors started discussing questions about gaps in the length spectra in 2005 when both were participating in the work of the thematic program "Time at work" at the Institut Henri Poincaré in 2005.

The authors thank D. Popov for stimulating their interest in this problem, and Y. Yomdin and V. Kaloshin for discussions related to Sections 5 of the present paper. The authors would also like to thank A. Glutsuk, N. Kamran, A. Katok, P. Sarnak and L. Silberman for stimulating discussions.

References

[Abr] R. Abraham. *Bumpy metrics* Global Analysis, Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc, Providence, R. I., 1970, 1–3.

[ABRS] M. Aka, E. Breuillard, L. Rosenzweig, N. de Saxcé. *Diophantine properties of nilpotent Lie groups.* Compos. Math. 151 (2015), no. 6, 1157–1188; arXiv:1307:1489v2

[A1] D. Anosov. *Geodesic flows on closed Riemannian manifolds of negative curvature.* Trudy Mat. Inst. Steklov. 90 (1967).

[A2] D. Anosov. *Generic properties of closed geodesics.* Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), no. 4, 675–709, 896.

[ArAv] V. Arnold, A. Avez. *Ergodic problems of classical mechanics*, W.A. Benjamin, Inc., 1968.

[B-W] A. Baker, G. Wustholz *Logarithmic forms and Diophantine geometry,* New Mathematical Monographs, 9. Cambridge University Press, Cambridge, 2007.

[Ban] V. Bangert *Mather sets for twist maps and geodesics on tori, Dynamics reported, Vol. 1, 1–56, Dynam. Report. Ser. Dynam. Systems Appl., 1, Wiley, Chichester, 1988.*

[B-S] Barreira L., Schmeling J. Sets of "non-typical" points have full topological entropy and full Hausdorff dimension.*Israel J. Math. 116 (2000), 29–70.*

[Bow] R. Bowen Equilibrium states and the ergodic theory of Anosov diffeomorphisms. *Lecture Notes in Math. 470. Springer, 1975.*

[B-R] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math. 29 (1975), no. 3, 181–202.*

[Br11] E. Breuillard. Heights on SL_2 and free subgroups. *Geometry, rigidity, and group actions, pp. 455–493, Chicago Lectures in Math., Univ. Chicago Press, 2011.*

[B-G] Yu A. Brudnyi and M. I. Ganzburg. A certain extremal problem for polynomials in n variables, *(Russian) Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973), 344–355.*

[D] D. Dolgopyat. Bounded orbits of Anosov flows. *Duke Math. J. 87 (1997) 87–114.*

[E] P. Eberlein. When is a geodesic flow of Anosov type? I, II. *Jour. Diff. Geometry 8 (1973), 437–463, 565–577.*

[F-LJ] J. Franchi and Y. Le Jan. Hyperbolic dynamics and Brownian motion. An introduction. *Oxford Math. Monographs. Oxford Univ. Press, Oxford, 2012.*

[GJS] A. Gamburd, D. Jakobson and P. Sarnak. Spectra of elements in the group ring of $SU(2)$. *Jour. of European Math. Soc. 1(1) (1999), 51–85.*

[G-R] H. Garland, M. S. Raghunathan Fundamental domains for lattices in $\mathbf{R}$-rank 1 semisimple Lie groups, *Ann. of Math. 92 (1970) 279–326.*

[Glu] A. Glutsyuk. Instability of nondiscrete free subgroups in Lie groups. *Transform. Groups 16 (2011), no. 2, 413–479.*

[Has] B. Hasselblatt. Hyperbolic dynamical systems. *Handbook of dynamical systems, Vol. 1A, 239–319, North-Holland, Amsterdam, 2002.*

[Hej] D. Hejhal. Selberg trace formula for $PSL(2,\mathbf{R})$, Vol. I. *Lecture Notes in Math. 548, Springer, 1976.*

[H-K] A. Katok and B. Hasselblatt. Introduction to the modern theory of dynamical systems. *Encyclopedia of Math. and its Apps, 54. Cambridge University Press, Cambridge, 1995.*

[JPT] D. Jakobson, I. Polterovich and J. Toth. Lower bounds for the remainder in Weyl's law on negatively curved surfaces. *IMRN 2007, article rnm142.*

[Kal] V. Yu . Kaloshin Growth rate of the number of periodic points, *in* Normal forms, bifurcations and finiteness problems in differential equations, *NATO Sci. Ser. II Math. Phys. Chem., 137 (2004) Kluwer, Dordrecht, 355–385.*

[K-R] *V. Kaloshin, I. Rodnianski* Diophantine properties of elements of SO(3). *Geom. Funct. Anal. 11 (2001), no. 5, 953–970.*

[K] *A. Katok* Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Inst. Hautes udes Sci. Publ. Math. No. 51 (1980), 137–173.*

[K-W] *G. Knieper, H. Weiss.* C^∞ genericity of positive topological entropy for geodesic flows on S^2, *J. Differential Geom. 62 (2002), no. 1, 127–141.*

[L-S] *W. Luo and P. Sarnak.* Number variance for arithmetic hyperbolic surfaces, *Comm. Math. Phys. 161 (1994), no. 2, 419–432.*

[Mar74] *G. A. Margulis.* Arithmetic properties of discrete subgroups. *Uspehi Mat. Nauk 29 (1974), no. 1 (175), 49–98.*

[Mar75] *G. A. Margulis.* Arithmeticity of nonuniform lattices in weakly noncompact groups. *Funkcional. Anal. i Prilozhen. 9 (1975), no. 1, 35–44.*

[Mar77] *G. Margulis.* Discrete groups of motions of manifolds of non-positive curvature. *Proceedings of the ICM (Vancouver, B.C., 1974), Vol. 2, pp. 21–34. Canad. Math. Congress, Montreal, Que., 1975.*

[Mar04] *G. Margulis.* On Some Aspects of the Theory of Anosov Systems *Translated from the Russian by V. V. Szulikowska. With a survey by Richard Sharp* Periodic Orbits of Hyperbolic Flows *Springer Monographs in Mathematics. Springer–Verlag, Berlin, 2004. vi+139 pp.*

[M-H] *J. C. Mason, Handscomb D. C.* Chebyshev polynomials. *Chapman & Hall/CRC, Boca Raton, FL, 2003. xiv+341 pp.*

[Miln] *J. Milnor.* A note on the curvature and fundamental group. *Jour. Diff. Geom. 2 (1967), 1–7.*

[Most] *G. Mostow.* Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. *IHES Publ. Math. 34 (1968), 53–104.*

[Par] *W. Parry.* Equilibrium states and weighted uniform distribution of closed orbits. *Dynamical Systems (College Park, MD 1986-87), Lecture Notes in Math. 1342, 617–625. Springer, 1988.*

[P-P] *W. Parry and M. Pollicott.* Zeta functions and closed orbit structure for hyperbolic systems. *Asterisque, 187-188 (1990), 1–256.*

[P-S] *V. Petkov and L. Stoyanov.* Distribution of periods of closed trajectories in exponentially shrinking intervals. *Comm. Math. Phys. 310 (2012), no. 3, 675–704.*

[P-R] *G. Prasad and A. Rapinchuk. Zariski-dense subgroups and transcendental number theory. Mathematical Rsearch Letters 12 (2005), 239–249.*

[Ran] *B. Randol.* The length spectrum of a Riemann surface is always of unbounded multiplicity. *Proceedings AMS 78 (1980), 455–456.*

[Ru] *D. Ruelle.* Resonances for axiom A flows. *J. Diff. Geom. 25 (1987), 99–116.*

[Sel] *A. Selberg.* On discontinuous groups in higher-dimensional symmetric spaces. *In* Contributions to function theory (Internat. Colloq. Function Theory, Bombay, 1960) *pp. 147–164. Tata Institute of Fundamental Research, Bombay.*

[Sin2] *Y. Sinai.* Gibbs measures in ergodic theory. *Uspehi Mat. Nauk 27 (1972), 4(166), 21–64. Translation: Russian Math. Surveys 27 (1972), 21–69.*

[Tak] *K.A. Takeuchi.* A characterization of arithmetic Fuchsian groups, *J. Math. Soc. Japan 27, No. 4 (1975), 600–612.*

[Thur] *W. P. Thurston.* Three-dimensional geometry and topology. Vol. 1. *Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.*

[vW] *B.L. van der Waerden B. L. Algebra. Vol. I. Springer, New York, 1991.*

[Var] *P. Varju.* Diophantine property in the group of affine transformation of the line. *Acta Sci. Math. (Szeged) 80 (2014), no. 3–4, 447–458.*

[Yom] *Y. Yomdin.* Remez–type inequality for discrete sets. *Israel J. Math. 186 (2011), 45–60.*

Department of Mathematics, Mathematics Building, University of Maryland, College Park, MD 20742-4015, USA

E-mail address: dmitry@math.umd.edu

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Str. West, Montréal QC H3A 2K6, Canada.

E-mail address: jakobson@math.mcgill.ca