Introduction to Numerical Analysis I

Spring Term 2006

Instructor: Georg Dolzmann

Final - May 17, 2006

You must write clearly and legibly. You must show all your work, no credit will be given for unsupported answers or results that we did not discuss in class. Open textbook (Atkinson only) and classnotes. No other references are allowed, in particular you are not allowed to discuss any aspects of the final with anybody else.

There are four problems on the final. All problems count 25 points. The final is due on the date and at the time of the final examination according to the official examination calendar.

The final is due on Wednesday, May 17, at 3:30PM! Please hand it in in my office, MATH 3309.

Problem 1: (a) Find the LU decomposition of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \\ 1 & 16 & 81 & 256 \end{pmatrix}$$

and solve the linear system $Ax = b$ where $b^T = (3, 1, -15, -107)$. Do not use a computer to do so and show all your work.
Check: All matrices and vectors are integer matrices and vectors!

(b) Suppose that a_i, $i = 1, \ldots, n$, is a basis of $\mathbb{R}^n$ and that we obtain a second basis if we replace the single vector a_k (k an index between 1 and n) by the vector $\widetilde{a}_k$ (i.e., the vector $\widetilde{a}_k$ is linearly independent of the vectors a_i, $i \neq k$). Moreover, assume that you know the coordinates of $\widetilde{a}_k$ with respect to the first basis, i.e., you know the vector $\widetilde{x} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ such that

$$\widetilde{a}_k = \sum_{i=1}^{n} \widetilde{x}_i a_i. \tag{1}$$

Suppose that b is a vector in $\mathbb{R}^n$. Explain how you can find the coordinates of b with respect to the second basis if you know the coordinates of b with respect to the first basis.

Hint: Recall that the coordinates $x = (x_1, \ldots, x_n)$ of the vector u in the basis a_i, i.e., the coefficients x_i in the representation

$$u = \sum_{i=1}^{n} x_i a_i,$$

are the solution of the linear system $Ax = u$, where A is the matrix which has the basis vectors a_i as columns.

(c) Suppose you want to solve a linear system $Ax = b$. After the computation of the LU decomposition of A you discover that one column in the matrix A was incorrect, say the kth column a_k. How can you find the solution z of the correct linear system

$$\widetilde{A}z = b \qquad \text{with } \widetilde{A} = (a_1, \ldots, a_{k-1}, \widetilde{a}_k, a_{k+1}, \ldots, a_n)$$

using the decomposition of the incorrect matrix, i.e., without computing a new LU decomposition for $\widetilde{A}$? Formulate an algorithm, perform an operation count (multiplications and divisions only), and compare with the work needed to solve a linear system.

(d) Implement this algorithm in MATLAB. Fill in the missing lines in the following m-file:

```
%
% subroutine for solving a linear system with A=LU
% given, but the k-th column in A is incorrect and
% the correct column is stored in akt. Output:
% the solution z in At z = b where At is the matrix
% obtained by replacing the kth column in A by akt.
%
%
% l   = lower triangular matrix in A=LU
% u   = upper triangular matrix in A=LU
% b   = right-hand side in the linear system
% akt = correct column
% k   = number of the incorrect column in A
% n   = dimension of the matrix
%

function z=smartsolve(l,u,b,akt,k,n)
```

Use the script `final06.m` which you can find on our class webpage to run your code. You are not allowed to change this file! Print the output generated by `final06.m` and print your file `smartsolve.m`. It must contain sufficiently many comments so that one can immediately understand what each line or your code does. You can implement the algorithm with less than 20 lines if you use vector operations whenever possible.

You can access the file `final06.m` directly at

`http://www.math.umd.edu/~dolzmann/Amsc466/final06.m`

Problem 2: Generalize to quadratic interpolation the material on the effect of rounding errors in table entries which we discussed for the linear case in class. Let $\epsilon_i = f(x_i) - \widetilde{f}_i$, $i = 0, 1, 2$, and

$$\epsilon = \max\left\{|\epsilon_i|,\ i = 1, 2, 3\right\}.$$

Show that the effect of these rounding errors on the quadratic interpolation error is bounded by 1.25ϵ, assuming that $x_0 \le x_1 \le x_2$ and $x_1 - x_0 = x_2 - x_1 = h$.

Problem 3: Derive the one and two-point Gaussian quadrature formulas for

$$I = \int_0^1 x f(x)\, \mathrm{d}x \sim \sum_{j=0}^{n} w_j f(x_j)$$

with the weight function $w(x) = x$ and $n = 0$ (one-point formula) and $n = 1$ (two point formula). You must show all your work.

Problem 4: Let $z > 0$ and define

$$y = \sqrt{z + \sqrt{z + \sqrt{z + \ldots}}}.$$

(a) Find a simple formula for y. Check your formula with $y = 3$ for $z = 6$.

(b) Define an iteration scheme for the computation of y by

$$y_{n+1} = \sqrt{z + y_n}\,, \qquad \text{with } y_0 > 0 \text{ given.}$$

Show that the scheme converges for all choices of y_0 with $0 \le y_0 \le y$.

(c) What is the order of convergence for this iteration scheme? In case that it is linear, what is the rate of convergence?

Hint: Here is a simple MATLAB code that computes the first values of the iteration with $z = 6$ and the output, including the iterates and the error $|y_n - y|$:

```
z=6;
max=10;
y=zeros(max,1);
y(1)=1;
format long

for i=2:max,
  y(i)=sqrt(z+y(i-1));
end;
y
error=abs(y-3)
```

Here is the MATLAB output:

```
      iterates                  error

  1.00000000000000      2.00000000000000
  2.64575131106459      0.35424868893541
```

```
2.94036584646615      0.05963415353385
2.99004445560031      0.00995554439969
2.99834028349024      0.00165971650976
2.99972336782748      0.00027663217252
2.99995389428362      0.00004610571638
2.99999231570410      0.00000768429590
2.99999871928374      0.00000128071626
2.99999978654728      0.00000021345272
```

Do these values confirm or contradict your theoretical predictions?