AMSC/CMCS 466 – Introduction to Numerical Analysis I

Spring Term 2006

Instructor: Georg Dolzmann

Homework set #11

Problem 1: Let $\mathcal{P}_2$ denote the space of all polynomials of degree less than or equal to two on the interval $[-1,1]$. Define the inner product on $\mathcal{P}_2$ by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)w(x)\,\mathrm{d}x$$

where the weight function w is given by $w(x) = 1 - x^2$. Use the Gram-Schmidt orthogonalization procedure to construct an orthogonal basis for $\mathcal{P}_2$ starting from the basis $\{p_0, p_1, p_2\} = \{1, x, x^2\}$.

Problem 2: The goal of the problem is to construct a quadrature scheme Q on the interval $[-1,1]$ to approximate the integral

$$I(f) = \int_{-1}^{1} f(x)\,\mathrm{d}x$$

with nodes $-1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$, i.e., to find a scheme that uses the end points of the interval as nodes. This quadrature scheme can be written as

$$Q(f) = \sum_{i=0}^{n} w_i f(x_i)$$

with suitable weights w_i.

a) Explain why it should be possible to find such a quadrature scheme that is exact on $\mathcal{P}_{2n-1}$, the space of all polynomials of degree less than or equal to $2n-1$.

b) Show that you can find the free nodes $x_1, \ldots, x_{n-1}$ as the zeros of a polynomial of degree $n-1$ that is orthogonal to all polynomials of degree less than or equal to $n-2$. Explain how you determine the weights w_i once you have found the nodes x_i.

Hint: Define the inner product as in Problem 2 by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)(1-x^2)\,\mathrm{d}x$$

and use the fact that you can write a polynomial p of degree less than or equal to $2n-1$ as $p(x) = z(x)q(x)(1-x^2) + r(x)$ if $q(x)$ is a polynomial of degree $n-1$ and where the degree of z is less than or equal to $n-2$ and the degree of r is less than or equal to n.

c) Find the quadrature scheme for $n = 3$, that is, find points $-1 = x_0 < x_1 < x_2 <$

$x_3 = 1$ and weights w_i, $i = 0, \ldots, 3$, such that

$$Q(f) = \sum_{i=0}^{3} w_i f(x_i)$$

is exact on all polynomials of degree less than or equal to five.