AMSC/CMCS 466 – Introduction to Numerical Analysis I

Spring Term 2006

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Homework set #7

Problem 1: [Atkinson 7-21, 7-22, 8-9, 7-11a] This problem deals with vector and matrix norms and facts from linear algebra.

(a) Prove the following: for $x \in \mathbb{R}^n$,

$$\begin{aligned}\|x\|_\infty &\le \|x\|_1 \le n\|x\|_\infty,\\ \|x\|_\infty &\le \|x\|_2 \le \sqrt{n}\|x\|_\infty,\\ \|x\|_2 &\le \|x\|_1 \le \sqrt{n}\|x\|_2.\end{aligned}$$

(b) Let A be a real nonsingular matrix of order n, and let $\|v\|_v$ denote a vector norm on $\mathbb{R}^n$. Define

$$\|x\|_* = \|Ax\|_v, \quad x \in \mathbb{R}^n.$$

Show that $\|\cdot\|_*$ is a vector norm in $\mathbb{R}^n$.

(c) Prove that if $A = LL^T$ with L real and nonsingular, then A is symmetric and positive definite.

(d) Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

(You may use MATLAB to check your answer, but you should do the calculations by hand.)

Problem 2: [Atkinson 8-21] The condition number $\mathrm{cond}(A)_*$ can be quite small for matrices A that are ill-conditioned. To see this, define the $n \times n$ matrix

$$A_R = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & -1 & \cdots & -1 \\ \vdots & & \ddots & & \vdots \\ & & & 1 & -1 \\ 0 & & \cdots & 0 & 1 \end{pmatrix}.$$

Easily $\mathrm{cond}(A)_* = 1$. Verify that A_n^{-1} is given by the upper triangular matrix $B = (b_{ij})$ with

$$b_{ii} = 1, \quad b_{ij} = 2^{j-i-1} \text{ for } i < j \le n.$$

Compute $\mathrm{cond}(A)_\infty$.

Problem 3: [Atkinson 8-13] As another approach to developing a compact method for producing the LU factorization of A, consider the following matrix oriented approach. Write

$$A = \begin{pmatrix} \widehat{A} & d \\ c^T & \alpha \end{pmatrix}, \quad c, d \in \mathbb{R}^{n-1}, \alpha \in \mathbb{R}$$

and $\widehat{A}$ square of order $n-1$. Assume A is nonsingular. Look for $A = LU$ in the form

$$\begin{pmatrix} \widehat{L} & 0 \\ m^T & 1 \end{pmatrix} \begin{pmatrix} \widehat{U} & q \\ 0 & \gamma \end{pmatrix}, \quad m, q \in \mathbb{R}^{n-1}, \gamma \in \mathbb{R}.$$

Show that m, q, and γ can be found, and describe how to do so. (This method is applied to an original A, factoring each principal submatrix in the upper left corner, in increasing order.

Remark: We are assuming here that A has an LU decomposition and that no pivoting is needed to find this decomposition.)

Problem 4: [Atkinson 8-18] (a) Calculate the condition numbers $\mathrm{cond}(A)_p$, $p = 1, 2, \infty$, for

$$A = \begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix}.$$

(b) Find the eigenvalues and eigenvectors of A, and use them to illustrate the remarks following (8.4.8) in Section 8.4. (Feel free to find the eigenvalues and eigenvectors with MATLAB, the numbers are not particularly nice.)