AMSC/CMCS 466 – Introduction to Numerical Analysis I

Spring Term 2006

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Homework set #8

Problem 1: (a) Suppose that A is a tridiagonal matrix, i.e.,

$$\begin{pmatrix} a_1 & c_1 & 0 & 0 & \dots & 0 \\ b_2 & a_2 & c_2 & 0 & & \\ 0 & b_3 & a_3 & c_3 & & \\ \vdots & & & \ddots & & \vdots \\ & & & b_{n-1} & a_{n-1} & c_{n-1} \\ 0 & \dots & & 0 & b_n & a_n \end{pmatrix}.$$

and that the LU decomposition of A can be found without pivoting. Show that A can then be decomposed as

$$A = LU = \begin{pmatrix} \alpha_1 & 0 & & \dots & 0 \\ b_2 & \alpha_2 & 0 & & \\ 0 & b_3 & \alpha_3 & & \\ \vdots & & & \ddots & \vdots \\ 0 & & \dots & b_n & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \gamma_1 & 0 & \dots & 0 \\ 0 & 1 & \gamma_2 & 0 & \\ \dots & & \ddots & & \dots \\ & & & 1 & \gamma_{n-1} \\ 0 & & \dots & 0 & 1 \end{pmatrix},$$

that is, the lower diagonal in A is equal to the lower diagonal in L. Verify that the coefficients α_i and γ_i are given the formulae

$$\alpha_1 = a_1, \quad \gamma_1 = \frac{c_1}{\alpha_1}$$

and

$$\alpha_i = a_i - b_i\gamma_{i-1} \text{ for } i = 2, \dots, n \quad \text{and} \quad \gamma_i = \frac{c_i}{\alpha_i} \text{ for } i = 2, \dots, n-1.$$

Hint: See the corresponding formulae in the text book!

Write a MATLAB routine `LUtridiag(A,b)` that computes the LU factorization of a tridiagonal matrix and solves the equation $Ax = f$. You can test your solver with matrices of the form

```
dim=10;      % choose your size
A=diag(rand(dim,1)-.5,0)+diag(rand(dim-1,1)-.5,-1) ...
  +diag(rand(dim-1,1)-.5,1);
A=A+2*diag(ones(dim,1),0);
f=rand(dim,1);
```

If you store L and U in the elements of A, then you can find the lower and the upper diagonal matrices with

```
L = tril(A,0);
[n,n] = size(A);
U = triu(A,1)+diag(ones(n,1));
```

Problem 2: The goal of this problem is to solve the boundary value problem

$$-u'' + u = f(x) \text{ in } (0,1), \quad u(0) = 0, \quad u(1) = 1$$

with

$$f(x) = \big(1 + \frac{\pi^2}{4}\big)\sin(\pi x/2) - \big(1 + 4\pi^2\big)\sin(2\pi x)$$

using a difference approximation. Note that the solution of this differential equation is given by

$$u(x) = \sin(\pi x/2) - \sin(2\pi x).$$

(a) In order to find a difference approximation, define a uniform partition of the interval $(0,1)$ into $(n+1)$ intervals (x_i, x_{i+1}) with length $h = 1/(n+1)$. Replace the second derivative $u''(x_i)$ in the points x_i, $i = 1, \ldots, n$, by a centered difference quotient (which corresponds to replacing the first derivative with a forward difference quotient and the second derivative with a backward difference quotient),

$$u''(x_i) \sim \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2}$$

and write down the equations that correspond to the differential equation. Use the boundary conditions $u(0) = 0$ and $u(1) = 1$ in the equations at the points x_1 and x_n, respectively.

(b) Find the linear system $Au = f$ that defines the solution at the points x_i, $i = 1, \ldots, n$, and show that the matrix A is positive definite. (Use problem set 6.) It is convenient to multiply the equations by h^2.

(c) Solve the linear system for $n = 8, 16, 32, 64, 128$ using `LUtridiag` and plot the discrete and the continuous solution. Make sure that you add the values of u at the boundary points before you plot the solution. Compute the error at the points x_i, and plot the result. Determine also

$$e_n = \Big(\sum_{i=1}^{n} h(u(x_i) - u_i)^2\Big)^{1/2}.$$

(Here $u(x_i)$ is the exact solution at the point x_i and u_i is the approximation on the computer at this point.) How does this error decrease as n increases? It corresponds to the integral of the square of the difference between the continuous solution and its approximation.