AMSC 612 – Numerical Methods for Partial Differential Equations

Spring Term 2004

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Homework set #9

Problem 1: The five-point formula leads to a discretization of second order for Laplace's equation. The goal of this problem is to find a formula of fourth order for the discretization of this equation.

(a) Identify the coefficients in the approximation of the second derivatives of a smooth function given by

$$u''(x) \sim D_h^2 u(x) = \frac{1}{h^2} \sum_{\nu=-2}^{2} c_\nu u(x + \nu h)$$

in such a way that the approximation is (at least) of order four, i.e.,

$$\sum_{\nu=-2}^{2} c_\nu u(x + \nu h) = u''(x) + \mathcal{O}(h^4).$$

(b) Use your scheme in (a) to write down a generalized star for the discretization of Laplace's equation. Why is this star not practical?

Problem 2: In this problem we experiment with the two ways to solve the Neumann problem for Laplace's equation on the unit square with the reduced system, that is, the versions with and without the corrections for the mismatch in the discrete compatibility condition. We want to solve

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = \phi \quad \text{on } \partial\Omega$$

with two sets of data,

$$u_1(x, y) = \frac{1}{2\pi^2} \sin(\pi x) \sin(\pi y) + \frac{1}{5\pi^2} \sin(\pi x) \sin(2\pi y)$$

with

$$f_1(x, y) = \sin(\pi x) \sin(\pi y) + \sin(\pi x) \sin(2\pi y), \tag{1}$$

and

$$u_2(x, y) = x^2 + y^2, \quad f_2(x, y) = -4. \tag{2}$$

We choose the Neumann data ϕ from the solution so that the (continuous) compatibility condition is always satisfied.

(a) Write a MATLAB code that solves the Neumann problem with the reduced system with and without the correction for the mismatch in the discrete compatibility condition

$$-h^2 \sum_{x\in\Omega_h} f(x) = h \sum_{x\in\Gamma'_h} \phi(x).$$

(b) Solve the two test examples with $J = 8, 16, 32, 64$, find the L^∞ errors, and determine the experimental rate of convergence. Here we define the L^∞ error as

$$\|u - u_h\|_\infty = \max_{x\in\Omega_h\cup\Gamma'_h} (u(x) - u_h(x)) - \min_{x\in\Omega_h\cup\Gamma'_h} (u(x) - u_h(x)).$$

The reason for this choice is that the solution is only determined up to a constant and therefore we need to factor out this arbitrary constant. Plot the error for $J = 32$ for both schemes. You can either plot the error only on the interior nodes Ω_h or you can define the solution u_h in the corners by interpolation and plot the error on $\Omega_h \cup \Gamma_h$. This can lead to artificial singularities of the error in the corners. Make also a double-logarithmic plot that shows both the error in the scheme without and the scheme with the correction.

Some tips: If you define the boundary conditions, the right-hand side, and the exact solution as `mfiles` or as inline functions, remember to make them "array smart". Use the commands `@phi` and `feval(phi,x,y)` to pass `phi` as an argument to a subroutine if `phi` is an `mfile`. If you choose a point which you eliminate from the system, make sure that the point lies always in Ω_h and not on Γ_h. Check that the scaling of L_h and of the terms in the right-hand side q_h that correspond to f and ϕ are correct. Use the discretization of the Neumann data find the values of the solution on Γ'_h.

You may want to solve this problem with various choices of data, for example with the following functions,

(3) $$u_3(x,y) = \frac{1}{2\pi^2}\cos(\pi x)\cos(\pi y), \quad f_3(x,y) = \cos(\pi x)\cos(\pi y).$$

The advantage of first working with (3) is that the Neumann conditions are zero and this allows us to test the part of the code that involves only the right-hand side f, the solution of the linear system and the representation of the solution. Also the effect of the discrete mismatch is minimal.