AMSC 612 – Numerical Methods for Partial Differential Equations

Spring Term 2004

Instructor: Georg Dolzmann

Homework set #3

Problem 1: [Morton&Meyers 2.7]
Show that the leading order term in the truncation error of the explicit scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{\big(U_{j+1}^n - U_j^n\big)p_{j+1/2} - \big(U_j^n - U_{j-1}^n\big)p_{j-1/2}}{(\Delta x)^2}$$

for the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(p(x)\frac{\partial u}{\partial x}\Big)$$

on the region $0 < x < 1$, $t > 0$ is given by

$$T_j^n = \frac{1}{2}\,\Delta t\, u_{tt} - \frac{1}{24}\,(\Delta x)^2\Big(\frac{\partial}{\partial x}\big(p\frac{\partial^3 u}{\partial x^3}\big) + \frac{\partial^3}{\partial x^3}\big(p\frac{\partial u}{\partial x}\big)\Big).$$

Here suitable boundary conditions are given at $x = 0$ and $x = 1$ and initial conditions are prescribed for $t = 0$. Deduce a bound on the global error of the result in terms of bounds on the derivatives of u and p, under the condition $0 < p(x)\Delta t \le \frac{1}{2}(\Delta x)^2$.

Hint: In order to expand the right hand side, expand the first divided differences at the points $x_{j+1/2}$ and $x_{j-1/2}$, respectively, and then expand the product of the result with p at the corresponding points. In this way you can avoid to multiply Taylor series for u and for p.

Problem 2: [Morton&Meyers 2.8]
Apply the θ-method to the problem of the previous exercise. Show that the maximum principle will apply provided that $2\nu(1-\theta)p(x) \le 1$.

Problem 3: Write a MATLAB code to implement the θ scheme given by

$$-\theta\nu U_{j-1}^{n+1} + (1 + 2\theta\nu)U_j^{n+1} - \theta U_{j+1}^{n+1} = \big(1 + (1-\theta)\nu\delta_x^2\big)U_j^n$$

to solve the one-dimensional heat equation

$$\begin{aligned} u_t &= u_{xx} \quad \text{in } (0,1)\times(0,\infty),\\ u(0,t) &= 0 \quad \text{for } t > 0,\\ u(1,t) &= 0 \quad \text{for } t > 0,\\ u(x,0) &= u^0(x) \quad \text{for } x \in (0,1), \end{aligned}$$

where

$$u^0(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Use $J = 20$, $\Delta x = 0.05$, $\Delta t = .01$ and choose three values for θ, namely $\theta = .25, .5, .75$. Plot the solutions for $n = 1, 10, 50$ and explain your observations.