AMSC 612 – Numerical Methods for Partial Differential Equations

Spring Term 2004

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Homework set #4

Problem 1: This problem set discusses how to solve the heat equation in a closed, heat conducting wire. Mathematically we have to solve the initial value problem for the heat equation $u_t = u_{xx}$ on the interval $[-1, 1]$ with periodic boundary conditions, $u(-1,t) = u(1,t)$, $u_x(-1,t) = u_x(1,t)$, and given initial conditions $u(x, 0) = u^0(x)$. We will solve the equation up to $t = 1$ in our numerical calculations.

(a) Find the general solution as a Fourier series using separation of variables.

(b) Verify that the Fourier series for the function

$$u^0(x) = \begin{cases} 0 & \text{for } |x| > \frac{1}{2}, \\ \frac{1}{2} & \text{for } |x| = \frac{1}{2}, \\ 1 & \text{for } |x| < \frac{1}{2}, \end{cases}$$

is given by

$$u^0(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\big((2k+1)\pi x\big).$$

Plot the partial sums of the Fourier series with 5, 11, 21, and 51 terms. What do you observe?

(c) Formulate an approximation of the heat equation with periodic boundary conditions with an explicit difference scheme. In order to do so, define as usual points $x_0 = -1 < x_1 < x_2 < \ldots < x_{J-1} < x_J = 1$ with $x_j - x_{j-1} = \Delta x$. Since we solve the heat equation for a closed wire, the points $x_0 = -1$ and $x_J = 1$ correspond to the same point on the wire and we can allow only one degree of freedom for this point. You also have to identify x_{J+1} with x_1 and x_{-1} with x_{J-1}. Write down the discrete form of the equation at the points x_0, x_j with $1 \leq j \leq J-2$, and at x_{J-1}.

(d) Define the total heat energy in the bar at time t_n by

$$H^n = \sum_{j=0}^{J-1} \Delta x U_j^n.$$

Show that the total heat is a conserved quantity, i.e., that $H^n = H^0$ for all $n \geq 0$.

(e) Implement the explicit scheme with $\nu = .4$ and $h = 1/10$, $1/20$, $1/40$, and $1/80$. Note that the exact solution is given by

$$u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} e^{-t(2k+1)^2\pi^2} \frac{(-1)^k}{2k+1} \cos\big((2k+1)\pi x\big)$$

and that you need only very few terms for $t = 1$ to get sufficient accuracy. Determine the experimental order of convergence in Δx (since $\nu = .4$ is fixed we can determine the order in Δx) in the L^2 norm,

$$e_2^n = \Big(\sum_{j=0}^{J-1} \Delta x \big(U_j^n - u(x_j, t_n)\big)^2\Big)^{1/2}.$$

Use

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dx=2/J;
x=[-1:dx:-.5-dx,-.5:dx:.5-dx,.5:dx:1-dx];
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to define the mesh points; in this way you can ensure that $-.5$ and $.5$ are mesh points and that the total energy in the discrete system is equal to the total energy in the continuous system (why?).