

AMSC 612 – Numerical Methods for Partial Differential Equations

Spring Term 2004

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Homework set #7

Problem 1: [Morton&Mayers, Problem 4.1]

Sketch the characteristics for the equation $u_t + au_x = 0$ for $0 < x < 1$ when $a(x) = x - \frac{1}{2}$. Set up the upwind scheme on a uniform mesh with $x_j = j\Delta x$, $j = 0, 1, \dots, J$, noting that no boundary conditions are needed, and derive an error bound; consider both even and odd J . Sketch the development of the solution when $u(x, 0) = x(1 - x)$ and obtain explicit error bounds by estimating the terms in the truncation error.

Repeat the exercise with $a(x) = \frac{1}{2} - x$, but with boundary conditions $u(0, t) = u(1, t) = 0$.

Problem 2: [Morton&Mayers 4.2]

If q has an expansion in powers of p of the form

$$q \sim c_1 p + c_2 p^2 + c_3 p^3 + c_4 p^4 + \dots,$$

show that

$$\tan^{-1} q \sim c_1 p + c_2 p^2 + \left(c_3 - \frac{1}{3} c_1^3\right) p^3 + \left(c_4 - c_1^2 c_2\right) p^4 + \dots$$

as in Lemma 4.1 in Section 4.4.

Use this result to derive the leading terms in the phase expansions of the following methods for approximating $u_t + au_x = 0$:

Upwind	$-\nu\xi + \frac{1}{6}\nu(1-\nu)(1-2\nu)\xi^3;$
Lax-Wendroff	$-\nu\xi + \frac{1}{6}\nu(1-\nu^2)\xi^3;$
Box	$-\nu\xi - \frac{1}{12}\nu(1-\nu^2)\xi^3;$
Leapfrog	$-\nu\xi + \frac{1}{6}\nu(1-\nu^2)\xi^3;$

where $\nu = a\Delta t/\Delta x$ and $\xi = k\Delta x$.

Problem 3: [Morton&Mayers, Problem 4.4]

Determine the coefficients c_0 , c_1 , and c_{-1} so that the scheme

$$U_j^{n+1} = c_{-1}U_{j-1}^n + c_0U_j^n + c_1U_{j+1}^n$$

for the solution of the equation $u_t + au_x = 0$ agrees with the Taylor series expansion of $u(x_j, t_{n+1})$ to as high an order as possible when a is a positive constant. Verify that the result is the Lax-Wendroff scheme.

In the same way, determine the constants in the scheme

$$U_j^{n+1} = d_1 U_{j-2}^n + c_{-1} U_{j-1}^n + c_0 U_j^n.$$

Verify that the coefficients d correspond to the coefficients c in the Lax-Wendroff scheme, but with ν replaced by $\nu - 1$. Explain why this is so, by making the change of variables $\xi = x - \lambda t$ in the differential equation where $\lambda = \Delta x \Delta t$. Hence, or otherwise, find the stability conditions for the scheme.