AMSC 612

Numerical Methods for Partial Differential Equations

Spring Term 2004

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Solutions to Homework Set 1

Problem 1: [Morton&Meyers 2.1]
(i) The function $u^0(x)$ is defined on $[0,1]$ by

$$u^0(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2-2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Show that the Fourier sine series of u^0 is given by

$$u^0(x) = \sum_{m=1}^{\infty} a_m \sin m\pi x$$

where

$$a_m = \frac{8}{m^2\pi^2} \sin\frac{m\pi}{2}.$$

(ii) Show that

$$\int_{2p}^{2p+2} \frac{1}{x^2}\,\mathrm{d}x \ge \frac{2}{(2p+1)^2}$$

and hence that

$$\sum_{p=p_0}^{\infty} \frac{1}{(2p+1)^2} < \frac{1}{4p_0}\,.$$

(iii) Deduce that $u^0(x)$ is approximated on the interval $[0,1]$ to within 0.001 by the sine series in part (i) truncated after $m = 405$.

Solution: (i) Note that $u_m(x) = \sqrt{2}\sin(m\pi x)$ is an orthonormal basis of $L^2(0,1)$ since

$$\int_0^1 \sin^2(m\pi x)\,\mathrm{d}x = \frac{1}{2}\int_0^1 \big(\sin^2(m\pi x) + \cos^2(m\pi x)\big)\,\mathrm{d}x = \frac{1}{2}.$$

The Fourier coefficients are therefore given by

$$\widetilde{a}_m = \sqrt{2}\int_0^1 u^0(x)\sin(m\pi x)\,\mathrm{d}x.$$

(We write $\widetilde{a}_m$ for the Fourier coefficient and a_m for the coefficient in the Fourier sine series.) By symmetry, the coefficient vanishes if u_m is an odd function with respect to $x = \frac{1}{2}$. The trig identities show that

$$\begin{aligned} \sin\big(m\pi(\frac{1}{2}+\xi)\big) &= \sin\frac{m\pi}{2}\,\cos m\pi\xi + \cos\frac{m\pi}{2}\,\sin m\pi\xi, \\ \sin\big(m\pi(\frac{1}{2}-\xi)\big) &= \sin\frac{m\pi}{2}\,\cos(-m\pi\xi) + \cos\frac{m\pi}{2}\,\sin(-m\pi\xi), \end{aligned}$$

and therefore u_m is an even function for odd m (the second term vanishes) and an odd function for even m (the first term vanishes). We only need to compute $\widetilde{a}_m$ for $m = 2k+1$ odd and we find

$$\begin{aligned} \widetilde{a}_m &= \int_0^1 u^0(x)\sqrt{2}\sin(m\pi x)\,\mathrm{d}x \\ &= 2\sqrt{2}\int_0^{1/2} 2x\sin(m\pi x)\,\mathrm{d}x \\ &= -2\sqrt{2}(2x)\,\frac{\cos(m\pi x)}{m\pi}\Big|_0^{1/2} + 2\sqrt{2}\int_0^{1/2} 2\,\frac{\cos(m\pi x)}{m\pi}\,\mathrm{d}x \\ &= 4\sqrt{2}\,\frac{\sin(m\pi x)}{(m\pi)^2} \\ &= \frac{4\sqrt{2}}{(m\pi)^2}\,\sin\frac{m\pi x}{2}. \end{aligned}$$

Note that $\sin(m\pi x/2)$ is equal to zero for m even, and this allows us to write the Fourier sine series as

$$u^0(x) = \sum_{m=1}^{\infty} \frac{8}{(m\pi)^2}\,\sin(m\pi x).$$

(ii) In order to prove the integral inequality we compute

$$\int_{2p}^{2p+2} \frac{1}{x^2}\,\mathrm{d}x = -\frac{1}{x}\Big|_{2p}^{2p+2} = \frac{1}{2p} - \frac{1}{2p+2} = \frac{2}{2p(2p+2)},$$

and thus we only need to check that $(2p+1)^2 \geq 2p(2p+2)$. This inequality is equivalent to $1 \geq 0$ and this establishes the estimate.

(iii) Since u^0 is a continuous and piecewise differentiable function, the sine series converges uniformly to u^0 and we only need to estimate the tails in the expansion. Let $m_0 = 2p_0 + 1$ be odd. Then

$$\Big|\sum_{m=m_0}^{\infty} \frac{8}{(m\pi)^2}\,\sin\frac{m\pi}{2}\,\sin(m\pi x)\Big| \leq \sum_{m=m_0}^{\infty} \frac{8}{(m\pi)^2} = \sum_{p=p_0}^{\infty} \frac{8}{\pi^2}\,\frac{1}{(2p+1)^2} \leq \frac{2}{\pi^2 p_0}.$$

This shows that the approximation error is less that 0.001 if $p_0 > 2000/\pi^2$ or $p_0 > 202.64$. To ensure this we have to choose $m_0 > 406.28$ and we conclude that the approximation is within the prescribed tolerance if the last term in the series corresponds to $m = 405$ (recall again that all even coefficients are equal to zero).

Problem 2: Write a MATLAB program to solve the heat equation

$$\begin{aligned} u_t &= u_{xx} && \text{in } (0,1) \times (0,\infty), \\ u(0,t) &= 0 && \text{for } t > 0, \\ u(1,t) &= 0 && \text{for } t > 0, \\ u(x,0) &= u^0(x) && \text{for } x \in (0,1), \end{aligned}$$

where

$$u^0(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Implement the explicit scheme that leads to

$$U_j^{n+1} = U_j^n + \nu\big(U_{j+1}^n - 2U_j^n + U_{j-1}^n\big)$$

for $j = 1, \ldots, J-1$, and $n \ge 1$. Here $\nu = \Delta t/(\Delta x)^2$ and

$$U_j^0 = u^0(x_j), \quad j = 1, 2, \ldots, J-1, \qquad U_0^n = U_J^n = 0, \quad n = 0, 1, 2, \ldots$$

Use $J = 20$, $\Delta x = 0.05$ and two values for Δt, namely $\Delta t = 0.0012$ and $\Delta t = 0.0013$. Plot the solution and the approximations with the finite difference schemes for $n = 0$, 1, 25, 50. Explain your observations in view of the stability analysis.

Solution: Here is the MATLAB code for the solution of the heat equation:

```
%
% function u=heateqn1d(u0,J,dt,nt)
%
% solves the heat equation u_t=u_{xx} on (0,1) with
% an explicit difference scheme
%
% J+1 nodes x_0,...,x_J in the x-variable, dx=1/J
%
% final time given by dt*nt
%
%
  function u=heateqn1d(u0,J,dt,nt)

  x=0:1/J:1;
  dx=1/J;
  nu=dt/(dx^2);


%
% un = solution at time n
%
% note that un has n+1 components labeled 1,...,n+1
%
%
```

```
un=u0(x);

for n=1:nt,
  un(2:J)=un(2:J)+nu*(un(3:J+1)-2*un(2:J)+un(1:J-1));
end;

u=un;
```

We use the function `heateqn1d` to generate the plots with the following commands:

```
close all
u0=inline('2*x.*(x>0).*(x<=.5)+2*(1-x).*(x>.5).*(x<1)');

J=20;

dt=0.0012;
x=0:1/J:1;
u=heateqn1d(u0,J,dt,0);

figure
plot(x,u,'g');
hold on
u=heateqn1d(u0,J,dt,1);
plot(x,u,'r')
u=heateqn1d(u0,J,dt,25);
plot(x,u,'b')
u=heateqn1d(u0,J,dt,50);
plot(x,u,'m')
title('explicit scheme, nu<1/2','FontSize',24)

dt=0.0013;
x=0:1/J:1;
u=heateqn1d(u0,J,dt,0);

figure
plot(x,u,'g');
hold on
u=heateqn1d(u0,J,dt,1);
plot(x,u,'r')
u=heateqn1d(u0,J,dt,25);
plot(x,u,'b')
u=heateqn1d(u0,J,dt,50);
plot(x,u,'m')
title('explicit scheme, nu>1/2','FontSize',24)
```

The results are shown in Figure 1.

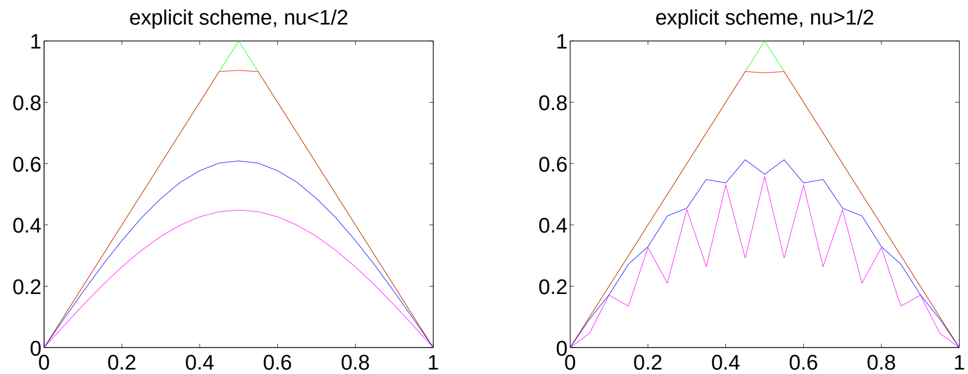

FIGURE 1. The solution of the heat equation computed with the explicit scheme. The parameters in the discretization are $\Delta x = 0.05$, $\Delta t = 0.0012$ (left panel) and $\Delta x = 0.05$, $\Delta t = 0.0013$ (right panel). The first choice of parameters corresponds to $\nu = \Delta t/(\Delta x)^2 = 12/25 < 1/2$ and the second choice to $\nu = 13/25 > 1/2$. The loss of stability of the scheme for $\nu > 1/2$ is clearly visible.