MATH 612

Numerical Methods for Partial Differential Equations

Spring Term 2004

Instructor: Georg Dolzmann

Solutions to Homework Set 7

Problem 1: [Morton&Mayers, Problem 4.1]

Sketch the characteristics for the equation $u_t + au_x = 0$ for $0 < x < 1$ when $a(x) = x - \frac{1}{2}$. Set up the upwind scheme on a uniform mesh with $x_j = j\Delta x$, $j = 0, 1, \ldots, J$, noting that no boundary conditions are needed, and derive an error bound; consider both even and odd J. Sketch the development of the solution when $u(x, 0) = x(1 - x)$ and obtain explicit error bounds by estimating the terms in the truncation error.

Repeat the exercise with $a(x) = \frac{1}{2} - x$, but with boundary conditions $u(0, t) = u(1, t) = 0$.

Solution: *Part 1:* $a(x) = x - \frac{1}{2}$. The characteristics are the solutions of the ODEs

$$\dot{x} = x - \frac{1}{2}, \quad x(0) = x_0.$$

For $x = \frac{1}{2}$ we find straight line parallel to the t-axis, for $x \neq \frac{1}{2}$ we can solve the ODE by separation of variables:

$$\frac{\mathrm{d}x}{x - 1/2} = \mathrm{d}t$$

and hence

$$\ln\bigl|x - \frac{1}{2}\bigr| = t + c.$$

If $x_0 < \frac{1}{2}$, then $\dot{x} < 0$ at $t = 0$ and thus x is a decreasing function. It follows that

$$\frac{1}{2} - x = ce^t \quad \Leftrightarrow \quad x(t) = \frac{1}{2} - ce^t.$$

The initial condition implies $c = \frac{1}{2} - x_0$ and thus

$$x(t) = \frac{1}{2} + \bigl(x_0 - \frac{1}{2}\bigr)x^t.$$

Similarly, if $x_0 < \frac{1}{2}$, then the solution is increasing and

$$x - \frac{1}{2} = ce^t \quad \Leftrightarrow \quad x(t) = ce^t + \frac{1}{2}.$$

The initial condition shows that $c = x_0 - \frac{1}{2}$ and that

$$x(t) = \frac{1}{2} + \bigl(x_0 - \frac{1}{2}\bigr)e^t.$$

Note that the same formula holds for $x = \frac{1}{2}$. In order to find a closed form for the solution, we need to find the characteristic through the point (x, t), i.e., we need to solve

$$(x,t) = \big(\frac{1}{2} + (x_0 - \frac{1}{2})e^t, t\big).$$

This leads to

$$x_0 = \frac{1}{2} + (x - \frac{1}{2})e^{-t}$$

and the solution is given by

$$u(x,t) = \Big(\frac{1}{2} + (x - \frac{1}{2})e^{-t}\Big)\Big(\frac{1}{2} - (x - \frac{1}{2})e^{-t}\Big) = \frac{1}{4} - (x - \frac{1}{2})^2 e^{-2t}.$$

The key in the upwind scheme is that we choose a backward difference quotient if the characteristic speed is positive and a forward difference quotient if the characteristic speed is negative. This leads to

$$\begin{aligned} \frac{U_j^{n+1} - U_j^n}{\Delta t} + a_j^n \frac{U_{j+1}^n - U_j^n}{\Delta x} &= 0, \quad j < \frac{J}{2} \\ \frac{U_j^{n+1} - U_j^n}{\Delta t} + a_j^n \frac{U_j^n - U_{j-1}^n}{\Delta x} &= 0, \quad j \geq \frac{J}{2}. \end{aligned}$$

For j even this leads to a node of the discretization at $x = \frac{1}{2}$ in which the characteristic speed is equal to zero. Since in this case the spatial difference quotient drops out, we find that $U_j^{n+1} = U_j^n$ as expected.

In order to derive the error bound we solve for U_j^{n+1} and get that

$$\begin{aligned} U_j^{n+1} &= (1+\nu)U_j^n - \nu U_{j+1}^n, \quad j < \frac{J}{2}, \\ U_j^{n+1} &= (1-\nu)U_j^n + \nu U_{j-1}^n, \quad j \geq \frac{J}{2}. \end{aligned}$$

Here $\nu_j = a_j^n \Delta t / \Delta x$. Note that ν_j changes its sign at $x = \frac{1}{2}$ so that the coefficients on the right hand side are non-negative if the CFL-condition $|\nu_j| \leq 1$ holds for all j. Assuming this, the error at the next time step is a convex combination of errors at the previous step, and we find that

$$E^n = \max_j |e_j^n| \leq t_F \max_n T^n,$$

where T^n is a bound on the truncation error at time step t_n. To find a bound on the truncation error, we compute for $j < J/2$ that

$$\begin{aligned} T_j^n &= \frac{u_j^{n+1} - u_j^n}{\Delta t} + a_j^n \frac{u_{j+1}^n - u_j^n}{\Delta x} \\ &= \big[u_t + \frac{1}{2}(\Delta t)u_{tt}\big]_j^n + \mathcal{O}\big((\Delta t)^2\big) + a_j^n \big[u_x + \frac{1}{2}(\Delta x)u_{xx}\big]_j^n + \mathcal{O}\big((\Delta t)^2\big) \\ &= \big[\frac{1}{2}(\Delta t)u_{tt} + \frac{1}{2}\big(x_j - \frac{1}{2}\big)(\Delta x)u_{xx}\big]_j^n + \mathcal{O}\big((\Delta t)^2 + (\Delta x)^2\big). \end{aligned}$$

In view of the PDE we see that the scheme is of first order in Δt and Δx. To find an explicit bound, we compute from the exact solution that

$$u_{xx} = -2e^{-2t}, \quad u_{tt} = 4\big(x - \frac{1}{2}\big)^2 e^{-2t},$$

and therefore $|u_{xx}| \leq 2$ and $|u_{tt}| \leq 1$. Hence

$$E^n \leq \frac{1}{2}\big(\Delta x + \Delta t\big)t_F.$$

We plot the characteristics and the evolution of the solution for several times with the following MATLAB code:

```
x=inline(’0.5+(x0-0.5)*exp(t)’,’t’,’x0’);
u=inline(’0.25-(x-0.5).^2*exp(-2*t)’,’x’,’t’);

x0_val=0:.1:1;
t=0:.1:1;
figure
hold on
for i=1:length(x0_val)
  plot(x(t,x0_val(i)),t);
end
tval=0:.2:1;
z=0:.01:1;
figure
hold on
for i=1:length(tval)
  plot(z,u(z,tval(i)));
end
```

The corresponding plots are show in Figure 1.

Part 2: $a(x) = \frac{1}{2} - x$. In this case the characteristic speed is positive for $x < 1/2$ and negative for $x > 1/2$. This means that the characteristic lines are incoming characteristics at the boundary and we need boundary values at $x = 0$ and $x = 1$. We can solve the characteristic equations as before and find that

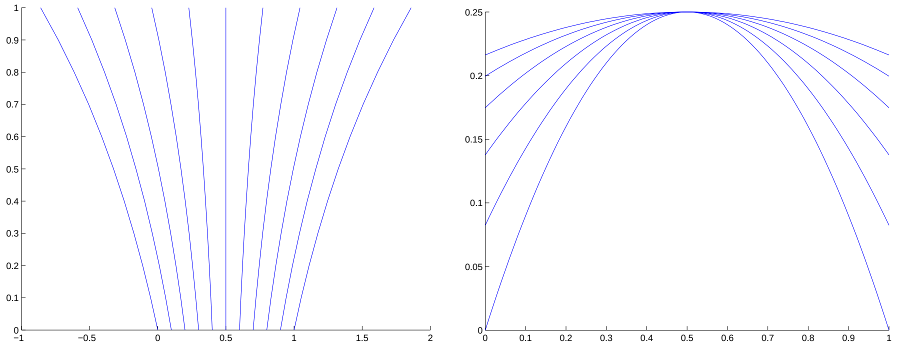

FIGURE 1. The characteristics and the evolution of the solution for $a(x) = x - \frac{1}{2}$. The characteristics spread out and the solution converges to a constant function.

$$x(t) = \frac{1}{2} + \big(x_0 - \frac{1}{2}\big)e^{-t}.$$

The exact solution is now given by

$$u(x,t) = \frac{1}{4} - \big(x - \frac{1}{2}\big)^2 e^{2t}.$$

The error estimate changes therefore in view of

$$u_{xx} = -2e^{2t}, \quad u_{tt} = 4\big(x - \frac{1}{2}\big)^2 e^{2t},$$

to $|u_{xx}| \le 2e^{2t}$ and $|u_{tt}| \le e^{2t}$. Hence

$$E^n \le \frac{1}{2}\big(\Delta x + \Delta t\big) e^{2t_F} t_F.$$

We plot the characteristics and the evolution of the solution with the following commands:

```
x=inline(’0.5+(x0-0.5)*exp(-t)’,’t’,’x0’);
u=inline(’max(0.25-(x-0.5).^2*exp(2*t),0)’,’x’,’t’);

x0_val=0:.1:1;
t=0:.1:1;
figure
hold on
for i=1:length(x0_val)
  plot(x(t,x0_val(i)),t);
end
tval=0:.2:1;
z=0:.001:1;
figure
hold on
for i=1:length(tval)
  plot(z,u(z,tval(i)));
end
```

The plots are show in Figure 2.

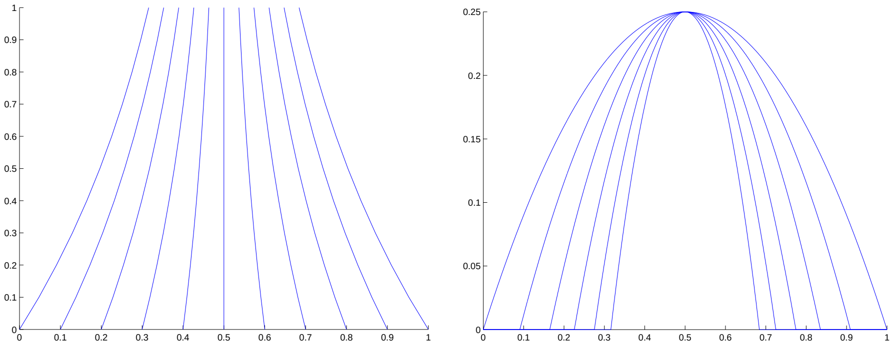

FIGURE 2. The characteristics and the evolution of the solution for $a(x) = \frac{1}{2} - x$. The characteristics focus and the solution slope of the solution increases in time. Note that the solution is continuous, but not differentiable along the characteristics emanating at the boundary points.

$$\begin{aligned} \frac{U_j^{n+1}-U_j^n}{\Delta t} + a_j^n\,\frac{U_j^n-U_{j-1}^n}{\Delta x} &= 0, \quad 0<j\le \frac{J}{2}, \\ \frac{U_j^{n+1}-U_j^n}{\Delta t} + a_j^n\,\frac{U_{j+1}^n-U_j^n}{\Delta x} &= 0, \quad \frac{J}{2}\le j< J. \end{aligned}$$

Problem 2: [Morton&Meyers 4.2]
If q has an expansion in powers of p of the form

$$q \sim c_1p + x_2p^2 + c_3p^3 + c_4p^4 + \ldots,$$

show that

$$\tan^{-1} q \sim c_1p + c_2p^2 + \big(c_3 - \frac{1}{3}\,c_1^3\big)p^3 + \big(c_4 - c_1^2c_2\big)p^4 + \ldots$$

as in Lemma 4.1 in Section 4.4.

Use this result to derive the leading terms in the phase expansions of the following methods for approximating $u_t + au_x = 0$:

Upwind	$-\nu\xi + \frac{1}{6}\,\nu(1-\nu)(1-2\nu)\xi^3;$
Lax-Wendroff	$-\nu\xi + \frac{1}{6}\,\nu(1-\nu^2)\xi^3;$
Box	$-\nu\xi - \frac{1}{12}\,\nu(1-\nu^2)\xi^3;$
Leapfrog	$-\nu\xi + \frac{1}{6}\,\nu(1-\nu^2)\xi^3;$

where $\nu = a\Delta t/\Delta x$ and $\xi = k\Delta x$.

Solution: The Taylor series for the inverse tangent is given by

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

and thus

$$\begin{aligned} \tan^{-1}(q) &= \big[c_1p + c_2p^2 + c_3p^3 + c_4p^4 + \ldots\big] \\ &\quad - \frac{1}{3}\,\big[c_1p + c_2p^2 + c_3p^3 + c_4p^4 + \ldots\big]^3 \\ &\quad + \frac{1}{5}\,\big[c_1p + c_2p^2 + c_3p^3 + c_4p^4 + \ldots\big]^5 + \ldots. \end{aligned}$$

Expanding the brackets, we find

$$\tan^{-1}(q) = c_1p + c_2p^2 + (c_1 - \frac{1}{3}c_1^3)p^3 + (c_4 - c_1^2c_2)p^4 + \ldots$$

The Upwind scheme. We assume that $a>0$ so that the upwind scheme is given by

$$\frac{U_j^{n+1}-U_j^n}{\Delta t} + a\,\frac{U_j^n-U_{j-1}^n}{\Delta x} = 0.$$

The amplification factors are given by

$$\lambda = 1 - \nu(1-e^{-i\xi}),$$

and the argument of this complex number is equal to

$$-\tan^{-1}\left\{\frac{\nu\sin\xi}{1-\nu(1-\cos\xi)}\right\}.$$

We now use the Taylor series for sin and cos and the expansion for the inverse tangent to find that

$$\begin{aligned}\arg\lambda &= -\tan^{-1}\left\{\frac{\nu(\xi-\frac{1}{6}\xi^3+ldots)}{1-\frac{1}{2}\nu\xi^2+\dots}\right\}\\ &= -\tan^{-1}\left\{\nu\xi(1-\frac{1}{6}\xi^2+\dots)(1+\frac{1}{2}\nu\xi^2+\dots\right\}\\ &= -\tan^{-1}\left\{\nu\xi[1+(\frac{1}{2}\nu-\frac{1}{6})\xi^2+\dots]\right\}\\ &= -\tan^{-1}\left\{\nu\xi+\nu(\frac{1}{2}\nu-\frac{1}{6})\xi^3+\dots\right\}\\ &\sim -\nu\xi+\frac{1}{6}\nu(\nu-1)(2\nu-1)\xi^3.\end{aligned}$$

Here we used $c_1=\nu$, $c_2=0$, and $c_3=\nu(\frac{1}{2}\nu-\frac{1}{6})$.

The Lax-Wendroff scheme. The amplification factor turned out to be

$$\lambda = 1-i\nu\sin\xi-2\nu^2\sin^2\frac{\xi}{2},$$

and hence

$$\arg\lambda = -\tan^{-1}\left\{\frac{\nu\sin\xi}{1-2\nu^2\sin^2\frac{\xi}{2}}\right\}$$

The expansion of $\sin^2\xi$ begins with

$$\sin^2\xi = \big(\xi-\frac{1}{6}\xi^3+\dots\big) = \xi^2-\frac{1}{3}\xi^4,$$

and hence

$$\begin{aligned}\arg\lambda &= -\tan^{-1}\left\{\frac{\nu(\xi-\frac{1}{6}\xi^3+\dots)}{1-(\frac{\nu^2}{2}\xi^2+\dots)}\right\}\\ &= -\tan^{-1}\left\{\nu(\xi-\frac{1}{6}\xi^3+\dots)(1+\frac{\nu^2}{2}\xi^2+\dots)\right\}\\ &= -\tan^{-1}\left\{\nu\xi+\nu(\frac{1}{2}\nu^2-\frac{1}{6})\xi^2+\dots\right\}\\ &\sim -\nu\xi+\frac{1}{6}\nu(1-\nu^2)\xi^3+\dots.\end{aligned}$$

The box scheme. We first rewrite the amplification factor as

$$\begin{aligned}\lambda &= \frac{\cos\xi/2-i\nu\sin\xi/2}{\cos\xi/2+i\nu\sin\xi/2}\\ &= \frac{\cos^2\xi/2-\nu^2\sin^2\xi/2-2i\nu\cos\xi/2\sin\xi/2}{\cos^2\xi/2+\nu^2\sin^2\xi/2}\\ &= \frac{(1+\nu^2)\cos^2\xi/2-\nu^2-i\nu\sin\xi}{\cos^2\xi/2+\nu^2\sin^2\xi/2}.\end{aligned}$$

We now use Taylor expansions to find that

$$\cos^2\frac{\xi}{2} = \big(1-\frac{\xi^2}{8}+\dots\big)^2 = 1-\frac{\xi^2}{4} = \dots$$

and

$$(1+\nu^2)\cos^2\frac{\xi}{2}-\nu^2 = 1-(1+\nu^2)\frac{\xi^2}{4}+\dots$$

This implies

$$\begin{aligned}
\arg\lambda &= -\tan^{-1}\Big\{\big[\nu(\xi-\frac{\xi^3}{6}+\dots)\big]\big[1+(1+\nu^2)\frac{\xi^2}{4}+\dots\big]\Big\} \\
&= -\tan^{-1}\Big\{\nu\xi+[\frac{\nu}{4}(1+\nu^2)-\frac{\nu}{6}]\xi^3+\dots\Big\} \\
&\sim -\Big\{\nu\xi+\Big[\frac{\nu(1+\nu^2)}{4}-\frac{\nu}{6}-\frac{\nu^3}{3}\Big]\xi^3\Big\} \\
&\sim -\nu\xi-\frac{1}{12}\,\nu(1-\nu^2)\xi^3.
\end{aligned}$$

The leap-frog scheme. The positive root corresponds to the physical solution,

$$\lambda = -i\nu\sin\xi+\sqrt{1-\nu^2\sin^2\xi}.$$

Since $\sqrt{1+x}\sim 1+x/2$ and $\sin^2\xi=\xi^2-\frac{1}{3}\xi^3+\dots$ we find

$$\begin{aligned}
\arg\lambda &= -\tan^{-1}\Big\{\frac{\nu\sin\xi}{1-\frac{1}{2}\nu^2\sin^2\xi}\Big\} \\
&= -\tan^{-1}\Big\{\nu\big(\xi-\frac{1}{6}\xi^3+\dots\big)\big(1+\frac{1}{2}\nu^2\xi^2+\dots\big) \\
&= -\tan^{-1}\Big\{\nu\xi-\frac{1}{6}\nu\xi^3+\frac{1}{2}\nu^3\xi^3+\dots\Big\} \\
&\sim -\Big\{\nu\xi+\Big[\big(-\frac{\nu}{6}+\frac{\nu^3}{2}\big)-\frac{\nu^3}{3}\Big]\xi^3\Big\} \\
&= -\nu\xi+\frac{1}{6}\,\nu(1-\nu^2)\xi^3
\end{aligned}$$

Problem 3: [Morton&Mayers, Problem 4.4]

Determine the coefficients c_0, c_1, and c_{-1} so that the scheme

$$U_j^{n+1}=c_{-1}U_{j-1}^n+c_0U_j^n+c_1U_{j+1}^n$$

for the solution of the equation $u_t+au_x=0$ agrees with the Taylor series expansion of $u(x_j,t_{n+1})$ to as high an order as possible when a is a positive constant. Verify that the result is the Lax-Wendroff scheme.

In the same way, determine the constants in the scheme

$$U_j^{n+1}=d_1U_{j-2}^n+c_{-1}U_{j-1}^n+c_0U_j^n.$$

Verify that the coefficients d correspond to the coefficients c in the Lax-Wendroff scheme, but with ν replaced by $\nu-1$. Explain why this is so, by making the change of variables $\xi=x-\lambda t$ in the differential equation where $\lambda=\Delta x\Delta t$. Hence, or otherwise, find the stability conditions for the scheme.

Solution: The Lax-Wendroff scheme can be written as

$$U_j^{n+1} = \frac{1}{2}\nu(\nu+1)U_{j-1}^n + (1-\nu^2)U_j^n + \frac{1}{2}\nu(\nu-1)U_{j+1}^n.$$

Note that $u_t = -au_x$, and therefore $u_{tt} = -au_{xt} = a^2u_{xx}$. We may thus replace t derivative by x derivatives if we multiply by $-a$. With $\nu = a\Delta t/\Delta x$ we find

$$\begin{aligned} U_j^{n+1} &= u + \Delta t\, u_t + \frac{1}{2}(\Delta t)^2 u_{tt} + \frac{1}{6}(\Delta t)^3 u_{ttt} + \dots \\ &= u - \nu\Delta x\, u_t + \frac{1}{2}\nu^2(\Delta x)^2 u_{tt} - \frac{1}{6}\nu^3(\Delta x)^3 u_{ttt} + \dots. \end{aligned}$$

The right-hand side of the scheme can be expanded as

$$\begin{aligned} RHS &= c_{-1}\big(u - \Delta x\, u_x + \frac{1}{2}(\Delta x)^2 u_{xx} - \frac{1}{6}(\Delta x)^3 u_{xxx} + \dots\big) \\ &+ c_0 u \\ &+ c_1\big(u + \Delta x\, u_x + \frac{1}{2}(\Delta x)^2 u_{xx} + \frac{1}{6}(\Delta x)^3 u_{xxx} + \dots\big). \end{aligned}$$

Comparing the powers of Δx we find the linear system

$$\begin{aligned} c_{-1} + c_0 + c_1 &= 1, \\ -c_{-1} + c_1 &= -\nu, \\ c_{-1} + c_1 &= \nu^2 \end{aligned}$$

which has the solution

$$c_{-1} = \frac{1}{2}\nu(\nu+1), \quad c_0 = 1-\nu^2, \quad c_1 = \frac{1}{2}\nu(\nu-1)$$

and this establishes the first assertion.

We now expand the right hand side for the second scheme by

$$\begin{aligned} RHS &= d_{-2}\big(u - 2\Delta x\, u_x + 2(\Delta x)^2 u_{xx} - \frac{4}{3}(\Delta x)^3 u_{xxx} + \dots\big) \\ &+ d_{-1}\big(u - \Delta x\, u_x + \frac{1}{2}(\Delta x)^2 u_{xx} - \frac{1}{6}(\Delta x)^3 u_{xxx} + \dots\big) \\ &+ d_0 u. \end{aligned}$$

This leads to the linear system

$$\begin{aligned} d_{-2} + d_{-1} + d_0 &= 1, \\ -2d_{-2} - d_{-1} &= -\nu, \\ 4d_{-2} + d_{-1} &= \nu^2 \end{aligned}$$

which has the solution

$$d_{-2} = \frac{1}{2}\nu(\nu-1), \quad d_{-1} = \nu(2-\nu), \quad d_0 = \frac{1}{2}(\nu-1)(\nu-2).$$

These are the coefficients in the Lax-Wendroff scheme if we replace ν by $\nu-1$. To see the reason for this, we change variables,

$$u(x,t) = v(\xi,\tau) = v(x-\lambda t, t)$$

where $\tau = t$. Then

$$u_t = -\lambda v_\xi + v_\tau, \quad u_x = v_\xi,$$

and hence

$$u_t + au_x = v_\tau + (a - \lambda)v_\xi$$

In these variables, we find that

$$\nu^* = (a - \lambda)\frac{\Delta t}{\Delta x} = \nu - 1,$$

and hence the second scheme corresponds to the Lax-Wendroff scheme in the new variables. It is stable if

$$|\nu^*| \leq 1 \quad \Leftrightarrow \quad 0 \leq \nu \leq 2.$$

Finally, the point (ξ_j, τ_n) is given by $(\xi_j, \tau_n) = (x_j - n\Delta x, t_n)$. Thus $(\xi_j, t_0) = (x_j, t_0)$ and in each time step the points are shifted by Δx to the left. This explains why in the new scheme we are referring to the points x_{j-2}, x_{j-1} and x_j.