Math 241 - Calculus I - Sections 02**

Instructor: Prof. Dolzmann

Spring Term 2002

Test 1 - Solutions

1. Determine which of the following limits exist (as a real number), which do not exist, and which are ∞ or $-\infty$. If the limit exists, evaluate it. Give reasons for your answers.

$$\text{a) } \lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{3-x} \qquad \text{b) } \lim_{x\to 0} \frac{\sin(5x)}{2x} \qquad \text{c) } \lim_{x\to\pi^+} \frac{x(x+\pi)}{\cos x - 1}.$$

Solution: a) We can factor the quotient and simplify,

$$\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{3-x} = \lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{(\sqrt{3}-\sqrt{x})(\sqrt{3}+\sqrt{x})} = \lim_{x\to 3} -\frac{1}{\sqrt{3}+\sqrt{x}} = -\frac{1}{2\sqrt{3}}.$$

b) We use the substitution rule with $y = 5x$ and get in view of

$$\lim_{y\to 0} \frac{\sin y}{y} = 1$$

that

$$\lim_{x\to 0} \frac{\sin(5x)}{2x} = \lim_{y\to 0} \frac{\sin(y)}{2(y/5)} = \frac{5}{2} \lim_{y\to 0} \frac{\sin(y)}{y} = \frac{5}{2}$$

c) Since $\cos(\pi) = -1$, the limit in the denominator is different from zero and we get from the quotient rule that

$$\lim_{x\to\pi^+} \frac{x(x+\pi)}{\cos x - 1} = \frac{\lim_{x\to\pi^+} x(x+\pi)}{\lim_{x\to\pi^+}(\cos x - 1)} = \frac{2\pi^2}{-2} = -\pi^2.$$

2. a) Let

$$f(x) = \begin{cases} x^2+1 & \text{if } x \geq 2, \\ 2x+b & \text{if } x < 2, \end{cases}$$

where b is a given number. What should b be in order to make f continuous at 2?

b) Find the vertical asymptotes (if any) of the graph of f where

$$f(x) = \frac{2x+2}{x^2-1}$$

and sketch the graph of f.

Solution: a) The function f is continuous at 2 if the left-hand and the right-hand limit at 2 are equal and if the common value of these limits is equal to the value of

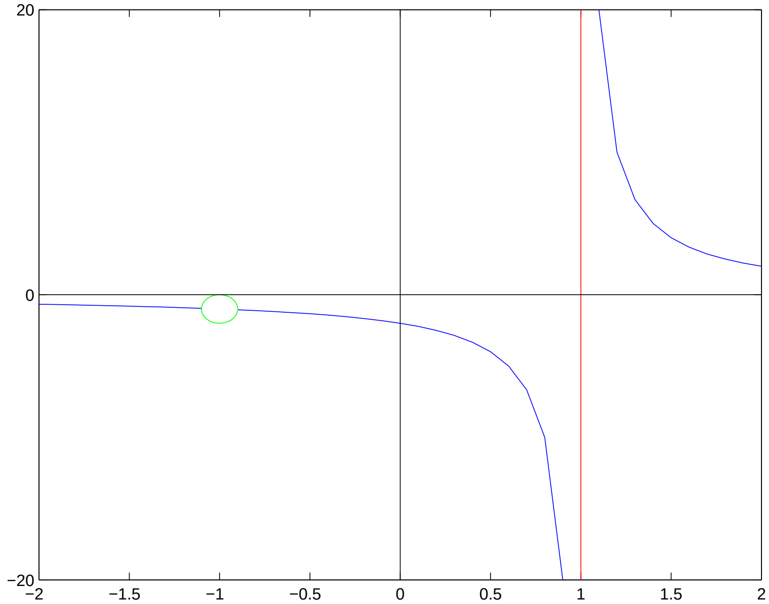

FIGURE 1. The graph of the function f in problem 2b). Note that f is not defined at $x = -1$ and $x = 1$.

f at 2. We have

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 + 1 = 5$$

and

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x + b) = 4 + b$$

and hence

$$5 = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 4 + b.$$

Thus equality holds if $b = 1$. Since $f(2) = 5$ the function is continuous with the choice $b = 1$.

b) Note that f is not defined for $x = 1$ and $x = -1$, but that

$$f(x) = \frac{2x+2}{x^2-1} = \frac{2(x+1)}{(x-1)(x+1)} = \frac{2}{x-1}.$$

We find that

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} = -1,$$

and thus f has a limit at $x = -1$. At $x = 1$ we have a vertical asymptote since

$$\lim_{x\to 1^+} f(x) = \infty, \qquad \lim_{x\to 1^-} = -\infty.$$

3. a) Let $f(x) = x^2 + 1$. Invoke the precise definition of the derivative to express $f'(-1)$ as a limit, and evaluate the limit.

b) Find an equation for the tangent line to the graph of $f(x) = x^2 + 1$ at the point $(-1, 2)$. Here you may use the fact that $f'(x) = 2x$.

Solution: We have

$$f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \to -1} \frac{x^2 + 1 - (1+1)}{x+1} = \lim_{x \to -1} x - 1 = -2.$$

The tangent line is thus given by

$$y = (-2)(x - (-1)) + 2 = -2x - 2.$$

4. Using the definition of limit (in terms of ϵ and δ), verify that

$$\lim_{x \to 2} -5x + 15 = 5.$$

Solution: Recall the definition of the limit: Let f be defined at each point of some open interval containing a, except possibly at a itself. Then a number L is the limit of $f(x)$ as x approaches a (or is the limit of f at a) if for every number $\epsilon > 0$ (output tolerance, give to us) there exists a number $\delta > 0$ (input tolerance, our friend) such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta.$$

We have to show that for all $\epsilon > 0$ we may choose $\delta > 0$ small enough such that

$$|f(x) - L| = |(-5x + 15) - 5| = |-5x + 10| = 5|x - 2| < \epsilon$$

provided that $|x - 2| < \delta$. In this case we choose $\delta = \epsilon/5$. Then

$$|f(x) - L| = 5|x - 2| < 5\delta = \epsilon.$$

Thus the limit of f at 2 is equal to 5.