Math 140 - Calculus I - Sections 02**

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Test 2 - Solutions

1. Find the following derivatives and simplify the resulting expressions if possible:

a) $f(x) = 5^x + x^5$, find $f'(x)$; **b**) $h(t) = \cos^4(e^{-t^2})$, find $h'(t)$;

c) $y = \ln(x^2+1)$, find $\dfrac{d^2y}{dx^2}$.

Solution: a) Since $5^x = \exp(x \ln 5)$ we find

$$f'(x) = (\ln 5)5^x + 5x^4.$$

b) In view of the chain rule,

$$h'(t) = 4\cos^3(e^{-t^2})\,\big(-\sin(e^{-t^2})\big)\,e^{-t^2}\,(-2t) = 8t\cos^3(e^{-t^2})\,\sin(e^{-t^2})\,e^{-t^2}.$$

c) Again by chain rule,

$$\frac{dy}{dx} = \frac{2x}{x^2+1},$$

and

$$\frac{d^2y}{dx^2} = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}.$$

2. a) Use implicit differentiation to find the derivative of y with respect to x at the point $(2,0)$, where y is implicitly given by

$$2e^{x^2y} = x.$$

b) Find an equation of the tangent line to the function y implicitly given by

$$\sin(x+y) = 2x \quad \text{at the point } (0,\pi).$$

Solution: a) We find by differentiating both sides with respect to x that

$$2e^{x^2y}\big(2xy + x^2\,\frac{dy}{dx}\big) = 1$$

and hence

$$\frac{dy}{dx} = \frac{1}{2x^2e^{x^2y}} - \frac{2y}{x}.$$

This formula gives at the point $(2,0)$ the value of the derivative for y to be equal to $1/8$.

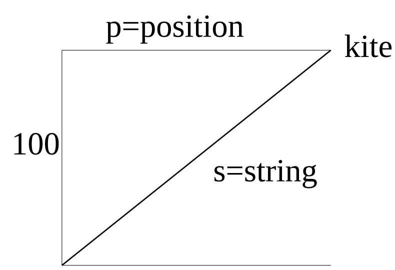

FIGURE 1. The variables in the kite problem.

b) We use again implicit differentiation to find

$$\cos(x+y)\big(1+\frac{dy}{dx}\big)=2,$$

and thus

$$\frac{dy}{dx}=\frac{2}{\cos(x+y)}-1.$$

Hence the slope of the tangent line is equal to -3 and thus

$$y=-3(x-0)+\pi=-3x+\pi$$

is the equation of the tangent line.

3. Use linear approximation to approximate the value of

$$\ln(0.95).$$

Solution: Recall that the formula for linear approximation is

$$f(a+h)\sim f(a)+hf'(a).$$

We choose $f(x)=\ln x$, $a=1$ and $h=-.05$. Then $f'(a)=1$ and

$$\ln(0.95)\sim 0+(-.05)=-.05.$$

4. A kite 100 feet above the ground is being blown away from the person holding its string. It moves in a direction parallel to the ground and at the rate of 10 feet per second. At what rate must the string be let out when the length of string already let out is 200 feet?

a) Make a sketch and label clearly the variables. State explicitly which rates are given and which rates you need to determine!

b) Which identity do you use to relate the variables?

c) Solve the problem.

Solution: a) Let s be the length of the string let out and let p define the position of the kite, i.e., p is the distance from the person holding the kite (on the ground), see Figure 1. In this notation, we know that

$$\frac{dp}{dt} = 10,$$

and we have to find

$$\left.\frac{ds}{dt}\right|_{t=t_0} \qquad \text{where } s(t_0) = 200.$$

b) We use that by the Pythagorean Theorem $s^2 = p^2 + 100^2$.

c) We differentiate this identity and find

$$2s\,\frac{ds}{dt} = 2p\,\frac{dp}{dt}$$

and hence

$$\frac{ds}{dt} = \frac{p}{s}\,\frac{dp}{dt}.$$

We find from the identity relating the variables that $p^2(t_0) = 200^2 - 100^2 = 3 \cdot 100^2$ and thus

$$\left.\frac{ds}{dt}\right|_{t=t_0} = \frac{\sqrt{3} \cdot 100 \cdot 10}{200} = 5\sqrt{3}.$$