Math 140 - Calculus I - Sections 02**

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Test 3 - Solutions

1. Determine whether the limits

$$\lim_{x\to\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x)$$

exist, do not exist, or are ∞ or $-\infty$ for the following functions:

$$\text{a) } f(x) = \frac{x^4+1}{x^2-1}, \qquad \text{b) } f(x) = x\cos x, \qquad \text{c) } f(x) = \frac{|x|+1}{x+1}.$$

Give reasons for your answers!

Solution: a) We have

$$\lim_{x\to\infty} \frac{x^4+1}{x^2-1} = \lim_{x\to\infty} \frac{x^2+1/x^2}{1-1/x^2} = \infty$$

since the limit of the numerator is ∞ and the limit of the denominator is equal to one. Similarly

$$\lim_{x\to-\infty} \frac{x^4+1}{x^2-1} = \lim_{x\to-\infty} \frac{x^2+1/x^2}{1-1/x^2} = \infty.$$

b) None of the limits exists. The limit can not be finite since the values of $x\cos x$ become arbitrarily large. On the other hand, the limit cannot be $\pm\infty$ since the values of $x\cos x$ oscillate between positive and negative values.

c) By the definition of the modulus,

$$\lim_{x\to\infty} \frac{|x|+1}{x+1} = \lim_{x\to\infty} \frac{x+1}{x+1} = 1$$

and

$$\lim_{x\to-\infty} \frac{|x|+1}{x+1} = \lim_{x\to-\infty} \frac{-x+1}{x+1} = \lim_{x\to-\infty} \frac{-1+1/x}{1+1/x} = -1.$$

2. Let $f(x) = x^2e^{-x}$.

a) Find all critical points of f and determine whether they correspond to maximum or minimum values of f.

b) Determine on which intervals f is increasing and on which intervals f is decreasing.

c) Determine on which intervals f is concave upwards and on which intervals f is concave downwards.

d) Sketch the graph of f. Make sure that your graph clearly reflects all the information you found in a) and b). Indicate clearly the behavior of f as x tends to $\pm\infty$.

Solution: a) We find

$$f'(x) = 2xe^{-x} + x^2(-e^{-x}) = (2x - x^2)e^{-x}$$

and

$$f''(x) = (2 - 2x)e^{-x} + (2x - x^2)(-e^{-x}) = (2 - 4x + x^2)e^{-x}.$$

The critical points are the zeros of the first derivative and thus given by

$$(2x - x^2)e^{-x} = x(2 - x)e^{-x} = 0.$$

There are two solutions, $x = 0$ and $x = 2$. Since

$$f''(0) = 2 > 0, \qquad f''(2) = -2e^{-2} < 0$$

we have a minimum at $x = 0$ and a maximum at $x = 2$.

b) Since f' changes its sign at the critical points, we have that f is increasing on $[0, 2]$ and that f is decreasing on $(-\infty, 0]$ and $[2, \infty)$.

c) We first find the inflection points as the zeros of the second derivative,

$$(2 - 4x + x^2)e^{-x} \quad \Leftrightarrow \quad (x - 2)^2 - 4 + 2 = 0 \quad \Leftrightarrow \quad x = 2 \pm \sqrt{2}.$$

Since e^{-x} is always positive, f'' changes its sign at these two points, and consequently these points are inflection points. We find

$$f''(x) > 0 \quad \text{on} \quad (-\infty, 2 - \sqrt{2}) \text{ and } (2 + \sqrt{2}, \infty)$$

and

$$f''(x) < 0 \quad \text{on} \quad (2 - \sqrt{2}, 2 + \sqrt{2}).$$

d) The graph of the function is shown in Figure 1. Note carefully that the limit as x tends to ∞ must be finite (and in fact is equal to zero), since the function is positive and does not have more than two extrema.

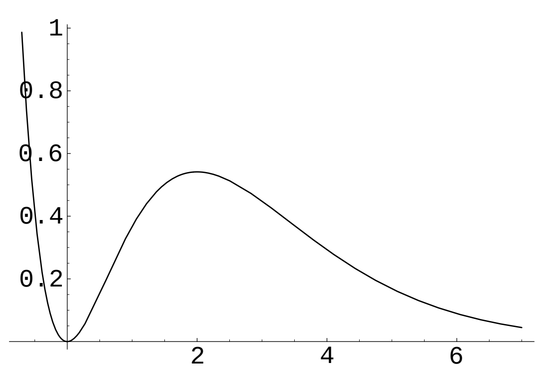

FIGURE 1. The graph of the function $f(x) = x^2e^{-x}$.

3. Suppose a window has the shape of a rectangle with an equilateral triangle attached at the top. Assuming that the perimeter of the window is 12 feet, find the dimensions that allow the maximum amount of light to enter.

Solution: The window is sketched in Figure 2. Since the perimeter of the window

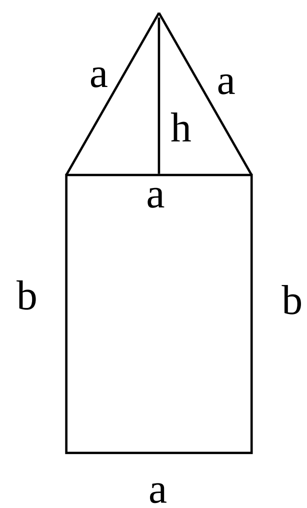

FIGURE 2. The geometry of the window.

is equal to 12 we have that

$$3a + 2b = 12 \quad \Leftrightarrow \quad b = 6 - \frac{3}{2}\,a.$$

The Pythagorean Theorem implies that

$$a^2 = h^2 + \left(\frac{a}{2}\right)^2 \quad \Leftrightarrow \quad h = \frac{\sqrt{3}}{2}\,a.$$

Thus the total area is equal to

$$A = ab + \frac{ah}{2} = a\left(6 - \frac{3}{2}\,a\right) + \frac{1}{2}\,a^2\,\frac{\sqrt{3}}{2}.$$

We find

$$\begin{aligned} A'(a) &= 6 - 3a + a\,\frac{\sqrt{3}}{2}, \\ A''(a) &= \left(-3 + \frac{\sqrt{3}}{2}\right) < 0, \end{aligned}$$

and thus any critical point corresponds to a maximum. There is only one solution of $A' = 0$ given by

$$12 = (6 - \sqrt{3})a \quad \Leftrightarrow \quad a = \frac{12}{6 - \sqrt{3}},$$

which gives thus the unique maximum value of the function. The corresponding value of b is given by

$$b = \frac{18 - 6\sqrt{3}}{6 - \sqrt{3}}.$$

4. Suppose you have a cache of radium, whose half-life is approximately 1590 years.

a) Find the constant k in the law $f(t) = f(0)e^{kt}$ for exponential decay.

b) How long would you have to wait for one tenth of the radium to disappear? Write down your answer in terms of logarithms.

Solution: a) We have

$$f(1590) = f(0)e^{1590k} = \frac{1}{2} f(0),$$

and hence

$$k = -\frac{\ln 2}{1590}.$$

b) If one tenth of the substance has disappeared, then $9/10$ remain, and we have to find t with

$$f(t) = f(0)e^{-\ln 2\, t/1590} = \frac{9}{10} f(0),$$

or

$$t = -1590 \frac{\ln 0.9}{\ln 2} \sim 241.7 \ \text{(years)}.$$