

Math 140 - Calculus I - Sections 02**

Instructor: Prof. Dolzmann

Fall Term 2002

Test 4 - Solutions

1. a) Let \mathcal{P} be a partition of the interval $[1, 3]$ and let $f(x) = (x - 2)^2$. Find the lower sum $L_f(\mathcal{P})$ and the upper sum $U_f(\mathcal{P})$!

b) Which is the larger number,

$$\int_1^2 (1+x) dx \quad \text{or} \quad \int_1^2 \sqrt{1+x^2} dx?$$

Justify your answer by a suitable comparison!

Solution: a) We sketch the graph of f in Figure 1. It follows that $M_1 = M_2 = 1$

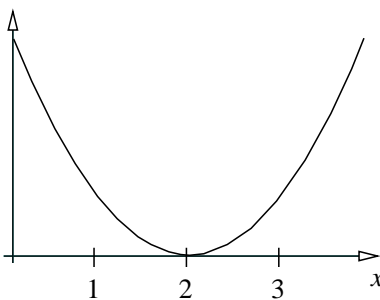


FIGURE 1. The graph of the function $f(x) = (x - 2)^2$.

and thus

$$U_f(\mathcal{P}) = \Delta x_1 M_1 + \Delta x_2 M_2 = 1 + 1 = 2.$$

Similarly, $m_1 = m_2 = 0$ and hence

$$L_f(\mathcal{P}) = \Delta x_1 m_1 + \Delta x_2 m_2 = 0 + 0 = 0.$$

b) Note that

$$1+x \geq \sqrt{1+x^2} \quad \Leftrightarrow \quad (1+x)^2 = 1+2x+x^2 \geq 1+x^2.$$

The last inequality is obviously true since $x \geq 0$ between the limits of the integral. Thus the general comparison principle implies that

$$\int_1^2 (1+x) dx \geq \int_1^2 \sqrt{1+x^2} dx.$$

2. a) Find the following integrals (simplify as much as possible and show all your work):

$$i) \int e^{6x} \sqrt{e^{3x} + 1} dx \quad ii) \int_1^3 |x - 2| dx$$

b) Evaluate the following expressions (show all your work, you need not simplify your answers):

$$iii) \frac{d}{dx} \int_{\sqrt{x}}^{e^x} t \csc t dt \quad iv) \int \frac{d}{dt} (\sec t + \tan t) dt.$$

Solution: i) We substitute $u = e^{3x} + 1$ and thus $du = 3e^{3x} dx$. It follows with $e^{3x} = u - 1$ that

$$\int e^{6x} \sqrt{e^{3x} + 1} dx = \frac{1}{3} \int (u - 1) \sqrt{u} du = \frac{2}{15} u^{5/2} - \frac{2}{9} u^{3/2} + C.$$

ii) Note that

$$|x - 2| = \begin{cases} 2 - x & \text{if } 1 \leq x \leq 2, \\ x - 2 & \text{if } 2 \leq x \leq 3. \end{cases}$$

Thus

$$\begin{aligned} \int_1^3 |x - 2| dx &= \int_1^2 (2 - x) dx + \int_2^3 (x - 2) dx \\ &= \left(2x - \frac{x^2}{2}\right) \Big|_1^2 + \left(\frac{x^2}{2} - 2x\right) \Big|_2^3 = \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

iii) By chain rule,

$$\frac{d}{dx} \int_{\sqrt{x}}^{e^x} t \csc t dt = (e^x \csc e^x) e^x - (\sqrt{x} \csc \sqrt{x}) \frac{1}{2\sqrt{x}}.$$

iv) By the fundamental theorem,

$$\int \frac{d}{dt} (\sec t + \tan t) dt = \sec t + \tan t + C$$

3. Find the area of the region in the plane that is bounded by the graphs of the functions $f(x) = \sin x$ and $g(x) = x - \pi$ on the interval $[0, 2\pi]$ on the x -axis, see Figure 2.

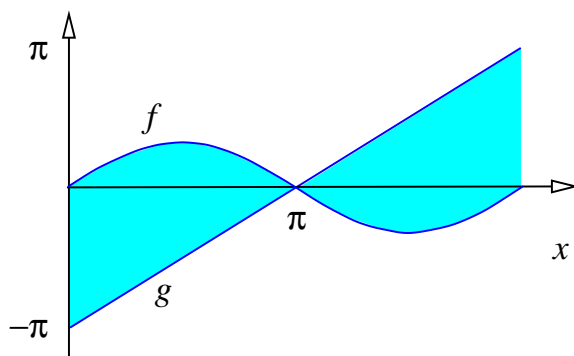


FIGURE 2. The region R in the plane.

Solution: Note that $f(x) \geq g(x)$ on $[0, \pi]$ and that $g(x) \geq f(x)$ on $[\pi, 2\pi]$. The area is thus given by

$$\begin{aligned}
 A &= \int_0^{2\pi} |f(x) - g(x)| \, dx \\
 &= \int_0^{\pi} (f(x) - g(x)) \, dx + \int_{\pi}^{2\pi} (g(x) - f(x)) \, dx \\
 &= \int_0^{\pi} (\sin x - (x - \pi)) \, dx + \int_{\pi}^{2\pi} (x - \pi - \sin x) \, dx \\
 &= \left(-\cos x - \frac{x^2}{2} + \pi x\right) \Big|_0^{\pi} + \left(\frac{x^2}{2} - \pi x + \cos x\right) \Big|_{\pi}^{2\pi} \\
 &= 4 + \pi^2.
 \end{aligned}$$

4. On the moon, the acceleration of an object in free-fall is $-1.6 \, m/s^2$. If a rock is thrown into a crater on the moon with an initial velocity of $-8 \, m/s$ and the rock hits the bottom 20 seconds later, what is the velocity at the time of the impact?

Solution: The velocity is just the integral of the acceleration. Let $v(0) = -8$ be the initial velocity. Then

$$v(t) - v(0) = \int_0^t a(s) \, ds = \int_0^t (-1.6) \, ds = -1.6t.$$

It follows that

$$v(20) = -8 - 1.6 \cdot 20 = -8 - 32 = -40,$$

and hence the velocity at the time of the impact is $-40 \, m/s$.