Math 241 - Calculus III - Section 1001

Spring Term 2002

Test 1 - February 25, 2002

Solutions

1. a) Find the area of the parallelogram spanned by the vectors

$$\boldsymbol{v}_1 = \boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{v}_2 = 2\boldsymbol{i} - \boldsymbol{j}.$$

b) What is the area of the triangle with corners

$$O = (0,0,0), \quad P_1 = (1,3,1), \quad P_2 = (2,-1,0)?$$

The area of the parallelogram is equal to the length of the cross product of the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$,

$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 1 \\ 2 & -1 & 0 \end{vmatrix} = \boldsymbol{i} + 2\boldsymbol{j} - 7\boldsymbol{k},$$

i.e.,

$$\text{area(parallelogram)} = \|\boldsymbol{v}_1 \times \boldsymbol{v}_2\| = \sqrt{1+4+49} = \sqrt{54} = 3\sqrt{6}.$$

The area of the triangle is half of the area of the parallelogram, i.e.,

$$\text{area(triangle)} = \frac{3}{2}\sqrt{6}.$$

2. Find an equation of the plane that contains the two lines ℓ_1 and ℓ_2 where ℓ_1 is given by the symmetric equation

$$\frac{x-2}{2} = \frac{y}{2}, \quad z = 1$$

and ℓ_2 is given by the parametric equation

$$x = -2t, \quad y = 2t + 2, \quad z = t + 2.$$

The plane contains the line vectors $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ of the two lines ℓ_1 and ℓ_2 and thus the normal vector $\boldsymbol{N}$ has to be perpendicular to these vectors.

We may therefore choose

$$\boldsymbol{N} = \boldsymbol{L}_1 \times \boldsymbol{L}_2 = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & 0 \\ 2 & 4 & 1 \end{vmatrix} = 2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}.$$

We get from the equation for the line ℓ_1 that $P_0 = (2, 0, 1)$ is on the line and thus in the plane. Hence an equation for the plane is given by

$$2(x-2) - 2y + 4(z-1) = 0.$$

3. Let $\boldsymbol{r}$ be given by

$$\boldsymbol{r}(t) = 2t\,\boldsymbol{i} + t^2\,\boldsymbol{j} + \frac{1}{3}t^3\,\boldsymbol{k}.$$

Find the length of the curve parameterized by $\boldsymbol{r}$ for $0 \le t \le 3$.

We have

$$\boldsymbol{r}'(t) = 2\boldsymbol{i} + 2t\boldsymbol{j} + t^2\boldsymbol{k}, \quad \|\boldsymbol{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} = 2 + t^2$$

Thus

$$\text{length} = \int_0^3 (2 + t^2)dt = 2t + \frac{t^3}{3}\Big|_0^3 = 15.$$

4. The position of a particle at time t is

$$\boldsymbol{r}(t) = 3\sin(2t)\,\boldsymbol{i} - 5\cos(2t)\,\boldsymbol{j} - 4\sin(2t)\,\boldsymbol{k}, \quad 0 \le t \le \pi.$$

Let C be the curve parameterized by $\boldsymbol{r}$.
a) Find the velocity, speed, and acceleration of the particle.
b) Find the tangential component of the acceleration.

We have

$$\begin{aligned} \boldsymbol{v}(t) &= 6\cos(2t)\,\boldsymbol{i} + 10\sin(2t)\,\boldsymbol{j} - 8\cos(2t)\,\boldsymbol{k}, \\ s(t) &= \sqrt{36\cos^2(2t) + 100\sin^2(2t) + 64\cos^2(2t)} = 10, \\ \boldsymbol{a}(t) &= -12\sin(2t)\,\boldsymbol{i} + 20\cos(2t)\,\boldsymbol{j} + 16\sin(2t)\,\boldsymbol{k}. \end{aligned}$$

Finally

$$\boldsymbol{v} \cdot \boldsymbol{a} = (-72 + 200 - 128)\sin(2t)\cos(2t) = 0,$$

and

$$a_{\boldsymbol{T}} = \frac{\boldsymbol{v} \cdot \boldsymbol{a}}{\|\boldsymbol{v}\|} = 0.$$

5. QB Kordell Stewart attempts a pass to his wide receiver. When the ball leaves his hand, it is 8 feet above the ground with an initial velocity $\boldsymbol{v}_0 = 32\boldsymbol{i} + 16\boldsymbol{k}$. The wide receiver misses and the ball touches the ground. How long was it in the air and how far did it travel (measured on the turf, from the feet of the QB to the point where it hit the turf)?

We set up a coordinate system such that the ball is flying along the x axis and such that the z axis is perpendicular to the field. The position of the ball is given by

$$\begin{aligned}\boldsymbol{r}(t) &= -\frac{1}{2}gt^2\boldsymbol{k} + \boldsymbol{v}_0 t + \boldsymbol{r}_0 \\ &= -16t^2\boldsymbol{k} + (32\boldsymbol{i} + 16\boldsymbol{k})t + 8\boldsymbol{k}.\end{aligned}$$

The ball hits the ground when the $\boldsymbol{k}$ component is equal to zero, i.e.,

$$-16t^2 + 16t + 8 = 0 \quad \Leftrightarrow \quad t^2 - t + \frac{1}{2} = 0.$$

Completing the square we find

$$\left(t - \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{2} = 0 \quad \Leftrightarrow \quad t = \frac{1}{2} \pm \frac{\sqrt{3}}{2}.$$

Since the time has to be positive, we ball flies for

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\,\text{sec}.$$

The distance traveled is given by the $\boldsymbol{i}$ component, i.e.

$$32\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = 16(1+\sqrt{3}) \sim 43.71\,\text{ft} \sim 14.57\,\text{yd}.$$

That's not a lot, is it?