Math 241 - Calculus III - Section 1001

Spring Term 2002

Test 2 - March 20, 2002

Solutions

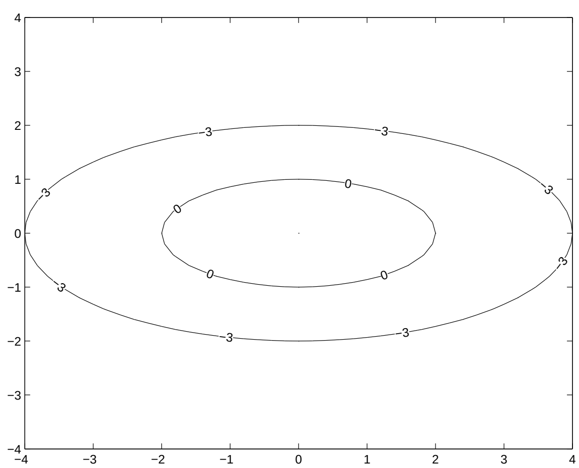
1. Let

$$f(x, y) = 1 - \frac{x^2}{4} - y^2.$$

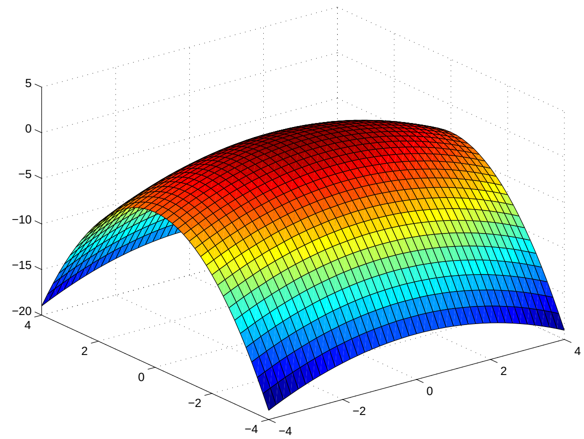
a) Sketch the level curves $f(x, y) = c$ for $c = -3$, 0, 1. Mark clearly the curves in your sketch.

b) Sketch the graph of f.

The level curve for $c = 1$ consists only of the origin, the others are ellipses:



The graph is an elliptic paraboloid:



You can generate these plots with the MATLAB commands

```
[X,Y]=meshgrid(-4:.2:4,-4:.2:4);
Z=1-X.^2/4-Y.^2;
surf(X,Y,Z)


[c,h]=contour(X,Y,Z,[-3 0 1]);
clabel(c,h)
```

2. Let

$$f(x,y) = xy + \frac{y^2}{3}.$$

At the point $(2,3)$ find

a) the directional derivative in the direction $\boldsymbol{a} = 3\boldsymbol{i} - \boldsymbol{j}$;

b) the direction of *maximal decrease* of f;

The gradient of f is given by

$$\operatorname{grad} f(x,y) = y\boldsymbol{i} + (x + \frac{2y}{3})\boldsymbol{j},$$

and thus $\operatorname{grad} f(2,3) = 3\boldsymbol{i} + 4\boldsymbol{j}$. Since $\|\boldsymbol{a}\| = \sqrt{10}$, $\boldsymbol{a}$ is not a unit vector, and we define

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{3}{\sqrt{10}}\boldsymbol{i} - \frac{1}{\sqrt{10}}\boldsymbol{j}.$$

The directional derivativ is then

$$D_{\boldsymbol{u}} f(2,3) = \text{grad}\, f(2,3) \cdot \boldsymbol{u} = \frac{5}{\sqrt{10}}$$

The direction of maximal decrease of the function is $-\text{grad}\, f(2,3) = -3\boldsymbol{i} - 4\boldsymbol{j}$.

3. Let $f(x,y) = \sqrt{x^2 + y^3}$.

a) Find the tangent plane to the graph of f at the point $(1,2,3)$.

b) Using your result in a), find an approximation for $f(1.04, 1.98)$. Your answer may be a sum of two or more terms, e.g., $5 - \frac{8}{11}$ instead of 4.2727.

We find

$$\text{grad}\, f(x,y) = \frac{x}{\sqrt{x^2+y^3}}\boldsymbol{i} + \frac{3y^2}{2\sqrt{x^2+y^3}}$$

The normal to the tangent plane is given by

$$\text{grad}\, f(1,2) - \boldsymbol{k} = \frac{1}{3}\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$$

and the equation of the tangent plane is thus

$$\frac{1}{3}(x-1) + 2(y-2) - (z-3) = 0.$$

To find the tangent plane approximation, we solve for z and obtain

$$z = 3 + \frac{1}{3}(x-1) + 2(y-2).$$

A good approximation is thus

$$\begin{aligned} z &= 3 + \frac{1}{3}(1.04 - 1) + 2(1.98 - 2) \\ &= 3 + \frac{1}{3}\frac{1}{25} - \frac{1}{25} = 3 - \frac{2}{75} \sim 2.9733. \end{aligned}$$

The correct value is 2.973885.

4. Let $f(x,y) = x^3 + y^3 - 6y^2 - 3x + 9$. Find and classify all critical points of f.

We have

$$\operatorname{grad} f(x,y) = (3x^2 - 3)\boldsymbol{i} + (3y^2 - 12y)\boldsymbol{j}$$

and thus the critical points $\operatorname{grad} f(x,y) = 0$ are given by

$$x^2 = 1, \quad \text{and} \quad 3y(y-4) = 0.$$

There are four critical points, $(1,0)$, $(-1,0)$, $(1,4)$, and $(-1,4)$. Since

$$D(x,y) = \left| \begin{pmatrix} 6x & 0 \\ 0 & 6y - 12 \end{pmatrix} \right| = 6x(6y-12),$$

we find

$$\begin{aligned} D(\ \ 1,0) &= -72 && \Rightarrow \text{saddle point,} \\ D(-1,0) &= \ \ \ 72 && \Rightarrow \text{relative extremum,} \\ D(\ \ 1,4) &= \ \ \ 72 && \Rightarrow \text{relative extremum,} \\ D(-1,4) &= -72 && \Rightarrow \text{saddle point.} \end{aligned}$$

Since $f_{xx}(-1,0) = -6$ and $f_{xx}(1,4) = 6$, we have a maximum at $(-1,0)$ and a minimum at $(1,4)$.

5. A factory produces Terp fan articles, namely sweat shirts and basketball caps. They calculate that their profit p depends on the number x of sweat shirts and y of basketball caps via the formula

$$p(x,y) = 80x^{3/4}y^{1/4}.$$

The production of one sweat shirt costs \$4 and of one baketball cap \$6. The total budget of the factory for the production is \$8000. The factory wants to maximize its profit. How many sweat shirts and basketball caps should they produce? Find the system of equations you get using Lagrangian multipliers. Do not solve the system!

The factory has only \$8000 for the production of the fan articles, thus the constraint is

$$g(x,y) = 4x + 6y = 8000.$$

The system of equations we have to solve, $\operatorname{grad} f(x,y) = \lambda \operatorname{grad} g(x,y)$ is thus

$$\begin{aligned} \frac{3}{4}\,80x^{-1/4}y^{1/4} &= 6\lambda, \\ \frac{1}{4}\,80x^{3/4}y^{-3/4} &= 4\lambda, \\ 4x + 6y &= 8000. \end{aligned}$$

Incidentially, it's not so difficult to solve the system. The first two equations are equal to

$$10x^{-1/4}y^{1/4} = \lambda, \quad 5x^{3/4}y^{-3/4} = \lambda,$$

and thus

$$10x^{-1/4}y^{1/4} = 5x^{3/4}y^{-3/4} \quad \Leftrightarrow \quad 2y = x.$$

The constraint now gives

$$4x + 6y = 12y = 8000 \quad \Leftrightarrow \quad y = 500$$

and then $x = 1000$ follows. Since we can only produce positive quantities of sweat shirts and basketball caps, we have to find the extrema on the line segment $4x + 6y = 8000$ in the first quadrant. Taking $x = 0$ and $y = 0$ gives minima, and $x = 1000$ and $y = 500$ gives the maximum value.