Math 241 - Calculus III - Section 1001

Spring Term 2002

Test 3 - April 19, 2002

Solutions

1. Let R be the region in the xy plane bounded by the lines $x = -1$, $y = 0$, $y = 1$ and the quarter circle $x^2 + y^2 = 1$ with $x \geq 0$, $y \geq 0$, see Figure 1. Find

$$\iint_R y\,dA.$$

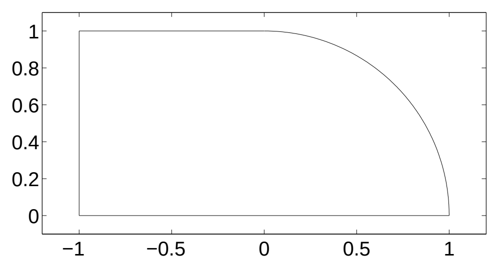

FIGURE 1. The region R in Problem 1

The region R is both vertically and horizontally simple, but it is easier to set up the integral on a horizontally simple region with $h_1(x) = -1$ and $h_2(y) = \sqrt{1-y^2}$. We find

$$\begin{aligned}\iint_R y\,dA &= \int_0^1 \int_{-1}^{\sqrt{1-y^2}} y\,dx\,dy \\ &= \int_0^1 y\big(\sqrt{1-y^2}+1\big)dy \\ &= -\frac{1}{2}\frac{2}{3}(1-y^2)^{3/2} + \frac{1}{2}y^2\Big|_0^1 = \frac{5}{6}.\end{aligned}$$

2. Let R be the triangle bounded by the lines

$$y = \frac{1}{3}\,x, \quad y = 2x, \quad y = \frac{5}{2} - \frac{1}{2}\,x,$$

see Figure 2. Use the change of coordinates $x = 3u + v$, $y = u + 2v$ to find

$$\iint_R (2x - y)\,dA.$$

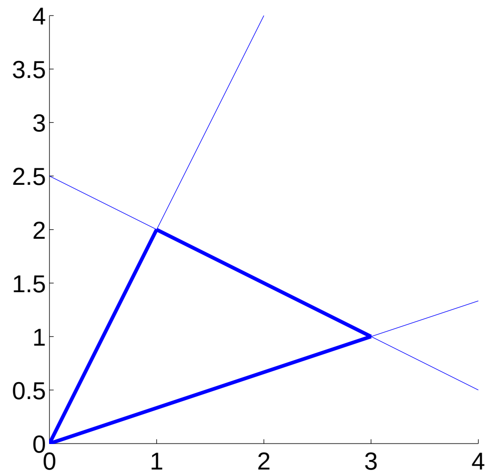

FIGURE 2. The region R in Problem 2

The first step, defining the coordinates, has been done in the problem. The second step is to express the equations defining R in the new coordinates. We find

$$y = \frac{1}{3}\,x \quad \Leftrightarrow \quad u + 2v = \frac{1}{3}\,(3u + v) \quad \Leftrightarrow \quad v = 0,$$

and

$$y = 2x \quad \Leftrightarrow \quad u + 2v = 2(3u + v) \quad \Leftrightarrow \quad u = 0,$$

and finally

$$y = \frac{5}{2} - \frac{1}{2}\,x \quad \Leftrightarrow \quad u + 2v = \frac{5}{2} - \frac{1}{2}\,(3u + v) \quad \Leftrightarrow \quad u + v = 1.$$

The third step is to find the Jacobian,

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5.$$

Finally, we set up the integral in u, v variables as

$$\begin{aligned}\iint_R (2x-y)dA &= \int_0^1 \int_0^{1-u} \big(2(3u+v)-(u+2v)\big)\, 5\, dv\, du \\ &= 25 \int_0^1 \int_0^{1-u} u\, dv\, du \\ &= 25 \int_0^1 (u-u^2)du = \frac{25}{6}.\end{aligned}$$

3. Let D be the solid region bounded below by the sphere $x^2+y^2+z^2 = 12$ and bounded above by the lower nappe of the cone $x^2+y^2 = 3z^2$, i.e., the surface $z = -\sqrt{(x^2+y^2)/3}$.

a) Sketch the solid region D and write clearly the equations for the lower nappe of the cone, the upper hemisphere and the lower hemisphere of the sphere in your sketch.

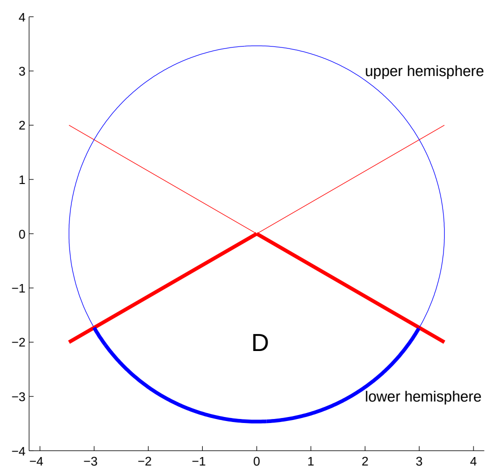

FIGURE 3. The intersection of the region D with the xz plane in Problem 3

b) Set up, but **do not evaluate** the integral

$$\iiint_D \big(x+|y|+z^2\big)dV$$

in rectangular, cylindrical, and spherical coordinates.

We need to find the region R in order to set up the integral in rectangular and cylindrical coordinates. We equate the equations for the sphere $z = -\sqrt{12 - x^2 - y^2}$ and the equation of the lower nappe of the cone to find

$$-\sqrt{12 - x^2 - y^2} = -\sqrt{(x^2 + y^2)/3}.$$

This is equivalent to

$$12 - x^2 - y^2 = (x^2 + y^2)/3 \quad \Leftrightarrow \quad 4(x^2 + y^2) = 36$$

or $x^2 + y^2 = 9$. The region R is therefore a circle with radius 3.

Rectangular coordinates: The region R is both vertically and horizontally simple. We set up the integral on a vertically simple region. The region R lies between the functions $g_1(x) = -\sqrt{9 - x^2}$ and $g_2(x) = \sqrt{9 - x^2}$. The region D lies above the sphere, i.e., $F_1(x, y) = -\sqrt{12 - x^2 - y^2}$, and below the lower nappe of the cone, i.e., $F_2(x, y) = -\sqrt{(x^2 + y^2)/3}$. If we denote the integral simply by I, then

$$I = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{12-x^2-y^2}}^{-\sqrt{(x^2+y^2)/3}} \left(x + |y| + z^2\right) dz\, dy\, dx.$$

Cylindrical coordinates: Cylindrical coordinates use polar coordinates in the plane, but leave z unchanged. In particular, $x^2 + y^2 = r^2$. Thus

$$I = \int_{0}^{2\pi} \int_{0}^{3} \int_{-\sqrt{12-r^2}}^{-r\sqrt{3}} \left(r\cos\theta + |r\sin\theta| + z^2\right) r\, dz\, dr\, d\theta.$$

Spherical coordinates: The equation of the sphere is $\rho^2 = 12$ and the equation of the cone becomes

$$\rho^2 \sin^2\phi = 3\rho^2\cos^2\phi \quad \Leftrightarrow \quad \tan^2\phi = 3.$$

This equation has two solutions in $(0, \pi)$, namely $\pi/3$ and $2\pi/3$. Note that $\phi = \pi/3$ corresponds to the upper nappe while $\phi = 2\pi/3$ gives the lower nappe of the cone. We find

$$I = \int_{0}^{2\pi} \int_{2\pi/3}^{\pi} \int_{0}^{\sqrt{12}} \left(\rho\cos\theta\sin\phi + |\rho\sin\theta\sin\phi| + \cos^2\phi\right) \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

4. Let R be the region in the first quadrant bounded by the circle $x^2 + y^2 = 2$, the parabola $y = x^2$ and the line $x = 0$, see Figure 4. Note that the circle and the parabola intersect at the point $(1,1)$. Use polar coordinates to find

$$\iint_R \frac{x^2}{(x^2+y^2)^{3/2}}\, dA.$$

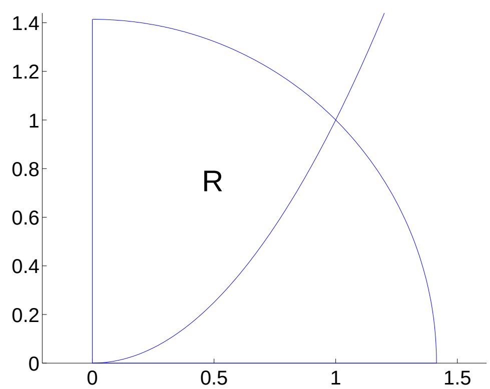

FIGURE 4. The region R in Problem 4

Useful trig identities

$$\begin{aligned}\cos^2\theta &= \frac{1}{2}(1+\cos 2\theta),\\ \sin^2\theta &= \frac{1}{2}(1-\cos 2\theta).\end{aligned}$$

Note that the point $(1,1)$ corresponds to $\theta = \pi/4$. It is therefore convenient to split the integral into two parts, one from 0 to $\pi/4$ and one from $\pi/4$ to $\pi/2$. We begin with the second integral:

$$\begin{aligned}\int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} \frac{r^2\cos^2\theta}{r^3}\, r\, dr\, d\theta &= \int_{\pi/4}^{\pi/2}\int_0^{\sqrt{2}} \cos^2\theta\, dr\, d\theta\\ &= \sqrt{2}\int_{\pi/4}^{\pi/2} \frac{1}{2}(1+\cos(2\theta))d\theta\\ &= \frac{\sqrt{2}}{2}\Big(\theta + \frac{1}{2}\sin(2\theta)\Big)\Big|_{\pi/4}^{\pi/2}\\ &= \frac{\sqrt{2}}{2}\Big(\frac{\pi}{4} - \frac{1}{2}\Big).\end{aligned}$$

For the second integral we use that in polar coordinates

$$y = x^2 \quad \Leftrightarrow \quad r \sin\theta = r^2 \cos^2\theta \quad \Leftrightarrow \quad r = \frac{\sin\theta}{\cos^2\theta}.$$

Thus

$$\begin{aligned}
\int_0^{\pi/4} \int_0^{\sin\theta/\cos^2\theta} \frac{r^2\cos^2\theta}{r^3}\, r\, dr\, d\theta &= \int_0^{\pi/4} \int_0^{\sin\theta/\cos^2\theta} \cos^2\theta\, dr\, d\theta \\
&= \int_0^{\pi/4} \cos^2\theta \frac{\sin\theta}{\cos^2\theta}\, d\theta \\
&= \int_0^{\pi/4} \sin\theta\, d\theta \\
&= -\cos\theta\Big|_0^{\pi/4} = 1 - \frac{1}{\sqrt{2}}.
\end{aligned}$$

The value of the integral is thus

$$\iint_R \frac{x^2}{(x^2+y^2)^{3/2}}\, dA = 1 + \frac{1}{\sqrt{2}}\left(\frac{\pi}{4} - \frac{3}{2}\right).$$