Math 241 - Calculus III - Section 1001

Spring Term 2002

Test 4 - May 13, 2002

Solutions

1. Evaluate the line integral

$$\int_{\mathcal{C}} x\,dx + yz\,dy + 2z^2\,dz,$$

where $\mathcal{C}$ is parameterized by

$$\boldsymbol{r}(t) = e^t\boldsymbol{i} + \sin t\boldsymbol{j} + \cos t\boldsymbol{k}, \quad 0 \le t \le \pi.$$

We use the definition of the line integral to find

$$\begin{aligned}
&\int_{\mathcal{C}} x\,dx + yz\,dy + 2z^2\,dz \\
&\quad = \int_0^{\pi} \big(e^t\,e^t + (\sin t)(\cos t)^2 + 2(\cos t)^2(-\sin t)\big)\,dt \\
&\quad = \int_0^{\pi} \big(e^{2t} - \sin t\cos^2 t\big)\,dt \\
&\quad = \frac{1}{2}\,e^{2t} + \frac{1}{3}\cos^3 t\Big|_0^{\pi} \\
&\quad = \frac{1}{2}\big(e^{2\pi} - 1\big) - \frac{2}{3}.
\end{aligned}$$

2. Find the line integral

$$\int_{\mathcal{C}} \boldsymbol{F}\cdot d\boldsymbol{r}$$

where

$$\boldsymbol{F}(x,y,z) = z^2\cos x\boldsymbol{i} + e^y\cos z\boldsymbol{j} + (2z\sin x - e^y\sin z)\boldsymbol{k},$$

and where $\mathcal{C}$ is parameterized by

$$\boldsymbol{r}(t) = \frac{\pi}{2}\,\sin(\frac{\pi}{2}\,t^4)\boldsymbol{i} + t^4\boldsymbol{j} + \frac{\pi}{2}\,\sin(\frac{\pi}{2}\,t^2)\boldsymbol{k}, \quad 0 \le t \le 1.$$

This integral it too complicated to be found explicitly by the definition. We use the fundamental theorem for line integrals instead, since

$$\operatorname{curl} \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ z^2 \cos x & e^y \cos z & 2z \sin x - e^y \sin z \end{vmatrix} = \mathbf{0}.$$

We need to find a potential f with $\operatorname{grad} f = \boldsymbol{F}$.

A. Solve $f_x(x,y,z) = M(x,y,z)$, i.e.,

$$f_x(x,y,z) = z^2 \cos x.$$

Integration in x gives

$$f(x,y,z) = z^2 \sin x + h(y,z)$$

with an unknown function $h(y,z)$.

B. We now solve $f_y(x,y,z) = N(x,y,z)$ and we use the information for f we found in the first step. Thus

$$f_y(x,y,z) = h_y(y,z) = e^y \cos z.$$

This is an equation for h, and an integration in y in

$$h_y(y,z) = e^y \cos z$$

yields

$$h(y,z) = e^y \cos z + g(z).$$

Thus

$$f(x,y,z) = z^2 \sin x + h(y,z) = z^2 \sin x + e^y \cos z + g(z).$$

C. We finally solve $f_z(x,y,z) = P(x,y,z)$ given the information from the second step. We find

$$f_z(x,y,z) = 2z \sin x - e^y \sin z + g'(z) = 2z \sin x - e^y \sin z.$$

Thus $g'(z) = 0$, i.e., $g(z) = c$ and the potential f is given by

$$f(x,y,z) = z^2 \sin x + h(y,z) = z^2 \sin x + e^y \cos z + c.$$

Now

$$\boldsymbol{r}(0) = \mathbf{0}, \quad \boldsymbol{r}(1) = \left(\frac{\pi}{2}, 1, \frac{\pi}{2}\right)$$

and thus

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = f\left(\frac{\pi}{2}, 1, \frac{\pi}{2}\right) - f(0,0,0) = \frac{\pi^2}{4} - 1.$$

3. Find

$$\iint_\Sigma z^2\, dS$$

where Σ is the part of the cone $z = \sqrt{x^2+y^2}$ between the planes $z = 1$ and $z = 2$.

We need to find the surface area element $dS = \sqrt{f_x^2 + f_y^2 + 1}\, dA$. Now

$$f_x(x,y) = \frac{x}{\sqrt{x^2+y^2}}, \quad f_y(x,y) = \frac{y}{\sqrt{x^2+y^2}},$$

and thus

$$f_x^2 + f_y^2 + 1 = \frac{x^2+y^2}{x^2+y^2} + 1 = 2, \quad dS = \sqrt{2}\, dA.$$

The region R lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. We find

$$\begin{aligned} \iint_\Sigma z^2\, dS &= \iint_R (x^2+y^2)\sqrt{2} dA \\ &= \sqrt{2} \int_0^{2\pi} \int_1^2 r^3\, dr\, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \frac{1}{4}(2^4 - 1^4) d\theta \\ &= \frac{15\sqrt{2}}{2}\pi. \end{aligned}$$

4. Let $\boldsymbol{F}(x,y,z) = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$, and let $\mathcal{C}$ be the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 4$. Orient $\mathcal{C}$ counterclockwise when viewed from above. Find

$$\int_\mathcal{C} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

The curve $\mathcal{C}$ forms the boundary of the surface Σ which is the part of the plane $x + y + z = 4$ that is contained in the cylinder. The orientation of $\mathcal{C}$ is induced by the orientation of Σ if the normal is pointing upwards. Moreover, Σ is the graph of $f(x,y) = z = 4 - x - y$ on the region R bounded by the circle $x^2 + y^2 = 1$. Since

$$\operatorname{curl} \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = -\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k},$$

we find by Stokes' Theorem that

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \iint_\Sigma \operatorname{curl}\boldsymbol{F}\cdot\boldsymbol{n}\,dS\\ &= \iint_R \big(-(-1)(-1)-(-1)(-1)+(-1)\big)dA\\ &= -3\iint_R 1\,dA = -3\pi.\end{aligned}$$

5. Let

$$\boldsymbol{F}(x,y,z) = (x+y^2)\boldsymbol{i} + (y+z^2)\boldsymbol{j} + (z+x^2)\boldsymbol{k}.$$

Find

$$\iint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS$$

where Σ is the boundary of the solid region bounded above by the plane $z = 2y$ and below by the paraboloid $z = x^2+y^2$ and where $\boldsymbol{n}$ is directed outward.

Useful formulae:

$$\begin{aligned}\int \sin^4\theta &= -\frac{1}{4}\sin^3\theta\cos\theta - \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta + C,\\ \int \cos^4\theta &= \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta + C.\end{aligned}$$

We find $\operatorname{div}\boldsymbol{F} = 3$ and thus by the Divergence Theorem

$$\iint_\Sigma \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iiint_D 3\,dV.$$

The solid lies between the graphs $z = x^2+y^2$ and $z = 2y$ and is parameterized on the region R given by

$$2y = x^2+y^2 \quad\Leftrightarrow\quad x^2+(y-1)^2 = 1.$$

It is convenient to introduce cylindrical coordinates, and the equation for R becomes

$$(r\cos\theta)^2 + (r\sin\theta-1)^2 = 1 \quad\Leftrightarrow\quad r^2 - 2r\sin\theta = 0.$$

Thus R is given as a polar graph by $r = 2\sin\theta$ with $0 \le \theta \le \pi$ since $r \ge 0$. Thus

$$\begin{aligned}
\iiint_D 3\,dV &= \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} 3\,r\,dz\,dr\,d\theta \\
&= 3\int_0^{\pi}\int_0^{2\sin\theta} (2r\sin\theta - r^2)\,r\,dr\,d\theta \\
&= 3\int_0^{\pi} \left(\frac{2}{3}\,r^3\sin\theta - \frac{1}{4}\,r^4\right)\Big|_0^{2\sin\theta} d\theta \\
&= 4\int_0^{\pi} \sin^4\theta\,d\theta \\
&= 4\Big(-\frac{1}{4}\,\sin^3\theta\cos\theta - \frac{3}{8}\,\sin\theta\cos\theta + \frac{3}{8}\,\theta\Big)\Big|_0^{\pi} \\
&= \frac{3}{2}\,\pi.
\end{aligned}$$