Math 461 - Linear Algebra for Scientists and Engineers

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Solutions

1. a) Find an example of a linearly independent set with three vectors in $\mathbb{R}^4$.

We can take for example

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_4 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ 0 \end{pmatrix} = \mathbf{0}$$

only if $c_1 = c_2 = c_3 = 0$.

b) Find an example of a linearly dependent set with two vectors in $\mathbb{R}^3$.

Take for example

$$\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Every set that contains the zero vector is linearly dependent.

c) For which values of the parameter a is the following set of vectors linearly independent?

$$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ a \\ 3 \end{pmatrix}.$$

We have to solve the linear system $A\mathbf{x} = 0$. Since

$$\begin{pmatrix} 2 & 4 & 2 \\ 1 & 2 & a \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 4 & 2 \\ 0 & 0 & 2a-2 \\ 0 & 1 & 3 \end{pmatrix}$$

we find that the homogeneous system has only the trivial solution if $a \neq 1$. Thus the vectors are linearly independent for $a \neq 1$.

2. Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{pmatrix}.$$

a) Find the reduced echelon form of A.

We have

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -4 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

b) Find the row space of A.

We have

$$\text{Row}\, A = \text{span}\{(1,0,-1),\ (0,1,2)\}.$$

c) Is the row space of A equal to the span of the first two rows in A? Justify your answer!

The answer is no. The first two rows of A span a one-dimensional subspace of $\mathbb{R}^3$, while $\text{Row}\, A$ is two-dimensional.

3. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix} \right\}$ be a basis of $\mathbb{R}^2$.

a) Find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^2$.

$$P_{\mathcal{B}} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$$

b) Find the $\mathcal{B}$-coordinates for the vectors $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$.

Since $[\,\mathbf{x}\,]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$ we have

$$[\,\mathbf{u}\,]_{\mathcal{B}} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}, \quad [\,\mathbf{v}\,]_{\mathcal{B}} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

4. Let A be given by

$$\begin{pmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The characteristic polynomial of A is given by

$$p(\lambda) = -(\lambda - 1)^2\,(\lambda - 2).$$

a) Find the eigenvalues of A and the corresponding eigenspaces.

Eigenvalues: $\lambda = 1$ (multiplicity 2) and $\lambda = 2$.

Eigenspaces: $\lambda = 1$. We have to solve $(A - I)\mathbf{x} = 0$.

$$A - I = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and therefore

$$\mathrm{Eig}(\lambda = 1) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$\lambda = 2$. We have to solve $(A - 2I)\mathbf{x} = 0$.

$$A - 2I = \begin{pmatrix} -1/2 & -1/2 & 0 \\ -1/2 & -1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and therefore

$$\mathrm{Eig}(\lambda = 1) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

b) Is A diagonalizable? If yes, find a diagonal matrix D and a matrix P such that $A = PDP^{-1}$.

We have a basis of eigenvectors, and thus A is diagonalizable with

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

5. Let A be the matrix

$$A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}.$$

a) Find the (complex) eigenvalues and corresponding eigenvectors for A.

The characteristic polynomial is $\det(A - \lambda I) = \lambda^2 + 4$ and the eigenvalues are thus $\lambda = 2i$ and $\lambda = -2i$.

Eigenspace for $\lambda = 2i$: solve $(A - 2iI)\mathbf{x} = 0$. Since

$$A - 2iI = \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \sim \begin{pmatrix} -2i & -4 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2i \\ 0 & 0 \end{pmatrix},$$

we have

$$\mathrm{Eig}(\lambda = 2i) = \mathrm{span}\left\{ \begin{pmatrix} 2i \\ 1 \end{pmatrix} \right\}, \ \mathrm{Eig}(\lambda = -2i) = \mathrm{span}\left\{ \begin{pmatrix} -2i \\ 1 \end{pmatrix} \right\}.$$

b) Is A similar to a matrix B of the form

$$B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

with a, $b \in \mathbb{R}$? If your answer is yes, find a matrix P and a matrix B such that $A = PBP^{-1}$. Give a geometric interpretation of B!

By our general results, $A = PBP^{-1}$ with

$$B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

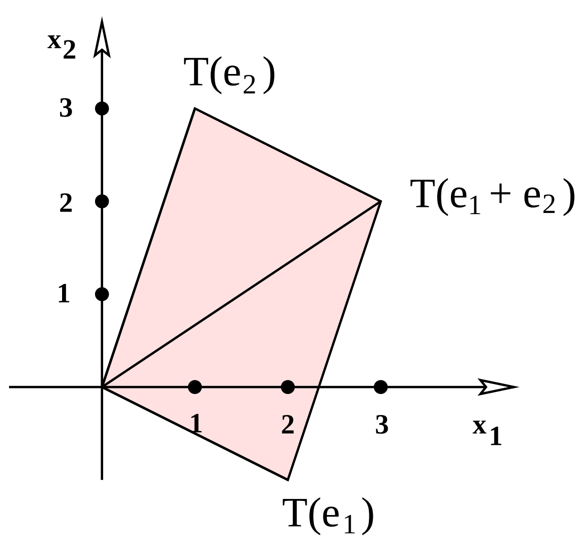
The matrix B is a scaling by $r = 2$ and a rotation through $\varphi = \pi/2$.

6. Let $T(\mathbf{x}) = A\mathbf{x}$ be the linear transformation given by

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}.$$

a) Plot the images of the unit vectors $\mathbf{e}_1 = (1,0)^T$, $\mathbf{e}_2 = (0,1)^T$ and of the vector $\mathbf{e}_1 + \mathbf{e}_2$. Sketch the image of the unit square Q.

$$T(\mathbf{e}_1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad T(\mathbf{e}_1 + \mathbf{e}_2) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



b) Is $T : \mathbb{R}^2 \to \mathbb{R}^2$ one-to-one?

Since $\det A = 7$, the system $Ax = b$ has a unique solution for all $b \in \mathbb{R}^2$ and T is one-to one.

c) Is T onto $\mathbb{R}^2$?

Since $\det A = 7$, the system $Ax = b$ has a unique solution for all $b \in \mathbb{R}^2$ and T maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.

d) What is the area of the image of the unit square Q under T?

$$\text{Area}(T(Q)) = (\det A)\text{Area}(Q) = 7.$$

7. Let

$$\mathbf{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

a) Find the orthogonal projection of $\mathbf{u}$ onto span$\{\mathbf{v}\}$.

$$\text{proj}_{\text{span}\{\mathbf{v}\}}(\mathbf{u}) = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{5}{2}\mathbf{v} = \begin{pmatrix} 5/2 \\ 5/2 \end{pmatrix}.$$

b) Is the $T(\mathbf{u}) = \text{proj}_{\text{span}\{\mathbf{v}\}}(\mathbf{u})$ a linear transformation? Justify your answer!

The answer is YES since

$$T(\mathbf{u}_1 + \mathbf{u}_2) = \frac{(\mathbf{u}_1 + \mathbf{u}_2)\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\mathbf{u}_1\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \frac{\mathbf{u}_2\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = T(\mathbf{u}_1) + T(\mathbf{u}_2),$$

and

$$T(c\mathbf{u}) = \frac{(c\mathbf{u})\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = c\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = cT(\mathbf{u}).$$

c) If your answer in b) is "YES", find the standard matrix for T!

The standard matrix has the columns

$$\mathbf{a}_1 = T(\mathbf{e}_1) = \frac{1}{2}\mathbf{v}, \quad \mathbf{a}_2 = T(\mathbf{e}_2) = \frac{1}{2}\mathbf{v}$$

and is given by

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

8. a) Describe all least-squares solutions of the equation $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}.$$

We have to solve the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$. We find

$$A^T A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \quad A^T b = \begin{pmatrix} 14 \\ 4 \\ 10 \end{pmatrix}.$$

We solve the linear system by

$$\begin{pmatrix} 4 & 2 & 2 & 14 \\ 2 & 2 & 0 & 4 \\ 2 & 0 & 2 & 10 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 & 2 & 14 \\ 0 & 2 & -2 & -6 \\ 0 & -2 & 2 & 6 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 & 4 & 20 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The general solution of this non-homogeneous system is given by

$$\mathbf{x} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

b) Recall that $A\widehat{x} = \widehat{b}$. Thus

$$\mathrm{proj}_{\mathrm{Col}\,A}(\mathbf{b}) = \widehat{\mathbf{b}} = \begin{pmatrix} 2 \\ 2 \\ 5 \\ 5 \end{pmatrix}$$