SOLUTION OF SECTION 3.3 PROBLEM 25

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25. Suppose f is a function that is continuous on a closed set F of real numbers. Show that f has a continuous extension to all of $\mathbf{R}$.

Solution:
The complement $\mathbf{R}\backslash F = \bigsqcup_{n>0} I_n$ is a countable union of disjoint open intervals. For $x \in I_n = (a_n, b_n)$, denote $a(x) = a_n$ and $b(x) = b_n$. Define $g : \overline{\mathbf{R}\backslash F} \longrightarrow \mathbf{R}$ by

$$g(x) = \begin{cases} f(a_n)\frac{b_n - x}{b_n - a_n} + f(b_n)\frac{x - a_n}{b_n - a_n} & \text{for } x \in I_n \\ f(x) \quad \text{for } x \in \partial(\mathbf{R}\backslash F) \subset F. \end{cases}$$

If g is continuous, then by the Pasting Lemma (See Munkres, *Topology*, 2^{nd} Edition, Chapter 2 §18, Constructing Continuous Functions, pp. 107-109) f and g combine to give a continuous function over all or $\mathbf{R}$. It remains to show that g is continuous. Obviously, it's continuous on $\mathbf{R}\backslash F$ since on each open interval it's the restriction of an affine function. For the remaining points, it's enough to show one-sided continuity. We'll do the argument for continuity on the left and continuity on the right follows *mutatis mutandis*. Let $(x_n)_{n>0} \subset \mathbf{R}\backslash F$ be a strictly increasing sequence converging to $x_\infty \in F$. Let $J_n = (a(x_n), b(x_n))$, then $J_n = I_{m_n}$. If $\{J_n\}_{n>0}$ is finite, then all but a finite number of elements in the sequence $(x_n)_{n>0}$ are in I_m for some $m > 0$ and necessarily $x_\infty = b_m$. Obviously, $\lim_{n\to\infty} g(x_n) = g(b_n) = g(x_\infty)$. Assume now that the family $\{J_n\}_{n>0}$ is infinite. Suppose also that all the J_n are all different, then $\sum_{n>0} l(J_n) \le x_\infty - a(x_1) < \infty$, so $\lim_{n\to\infty} l(J_n) = 0$. If the J_n are not all different, the sequence $(x_n)_{n>0}$ stays a finite time at each interval so still $\lim_{n\to\infty} l(J_n) = 0$. Since $a(x_n) < x_n < b(x_n) < x_\infty$, by the Squeeze lemma, $\lim_{n\to\infty} a(x_n) = \lim_{n\to\infty} b(x_n) = x_\infty$. Abusing notation, denote $a_n := a(x_n)$ and $b_n := b(x_n)$. Now,

$$\begin{aligned} |g(x_n) - g(x_\infty)| &= \left| f(a_n)\frac{b_n - x}{b_n - a_n} + f(b_n)\frac{x - a_n}{b_n - a_n} - f(x) \right| \\ &= \left| f(a_n)\frac{b_n - x}{b_n - a_n} + f(b_n)\frac{x - a_n}{b_n - a_n} - \left(f(x)\frac{b_n - x}{b_n - a_n} + f(x)\frac{x - a_n}{b_n - a_n} \right) \right| \\ &\le |f(a_n) - f(x)| \left| \frac{b_n - x}{b_n - a_n} \right| + |f(x) - f(b_n)| \left| \frac{x - a_n}{b_n - a_n} \right| \\ &\le |f(a_n) - f(x)| + |f(x) - f(b_n)|. \end{aligned}$$

The sequences $(a_n)_{n>0}$ and $(b_n)_{n>0}$ are in F, where f is continuous. This implies $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x_\infty)$, so by the inequalities above,

$$\lim_{n\to\infty} g(x_n) = g(x).$$

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