CHAPTER 3. THE COMPLETENESS THEOREM

1. Introduction

In this Chapter we prove Gödel's Completeness Theorem for first order logic.

Theorem 1.1. *(Completeness Theorem) Let $\Sigma \subseteq Sn_{\mathcal{L}}$.*

(a) *Σ is satisfiable iff Σ is consistent.*
(b) *For any $\theta \in Sn_{\mathcal{L}}$, $\Sigma \vdash \theta$ iff $\Sigma \models \theta$.*

By Soundness (Theorem 8.2 in Chapter 2) we know the left to right implications of both (a) and (b). Recall also Lemma 6.2 and Theorem 9.4 from Chapter 2 which assert that $\Sigma \models \theta$ iff $(\Sigma \cup \{\neg\theta\})$ is not satisfiable and that $\Sigma \vdash \theta$ iff $(\Sigma \cup \{\neg\theta\})$ is not consistent. Therefore part (b) of Theorem 1.1 follows from part (a). So it suffices the establish the following result.

Theorem 1.2. *(Model Existence) Let $\Sigma \subseteq Sn_{\mathcal{L}}$ be consistent. Then Σ is satisfiable, (i.e., Σ has a model).*

As in sentential logic the argument for Theorem 1.2 will involve maximal consistent sets of sentences (see below), but we will have to expand the original consistent set by "adding witnesses", a novel and important technique introduced by Leon Henkin in 1949.

The following definition is verbally the same as in sentential logic.

Definition 1.1. A set $\Gamma \subseteq Sn_{\mathcal{L}}$ is *maximal consistent* iff it is consistent and for every $\theta \in Sn_{\mathcal{L}}$ either $\theta \in \Gamma$ or $\neg\theta \in \Gamma$.

The lemma allowing us to extend consistent sets to maximal consistent sets is stated and proved exactly as in sentential logic.

Lemma 1.1. *Let $\Sigma \subseteq Sn_{\mathcal{L}}$ be consistent and let $\theta \in Sn_{\mathcal{L}}$. Then either $(\Sigma \cup \{\theta\})$ or $(\Sigma \cup \{\neg\theta\})$ is consistent.*

We next note that Finiteness is established exactly as for sentential logic.

Theorem 1.3. *Let $\Sigma \subseteq Sn_{\mathcal{L}}$.*

(a) *For any $\theta \in Sn_{\mathcal{L}}$, $\Sigma \vdash \theta$ iff there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \vdash \theta$.*
(b) *Σ is consistent iff every finite $\Sigma_0 \subseteq \sigma$ is consistent.*

Finally the result on the existence of maximal consistent sets, due to Lindenbaum, is also stated and proved just as in sentential logic.

Theorem 1.4. *Let $\Sigma \subseteq Sn_{\mathcal{L}}$ be consistent. Then there is some maximal consistent $\Gamma \subseteq Sn_{\mathcal{L}}$ such that $\Sigma \subseteq \Gamma$.*

Note that Γ is not usually uniquely determined by Σ. For example, if both $\Sigma \cup \{\theta\}$ and $\Sigma \cup \{\neg\theta\}$ are consistent, then there will be maximal consistent sets Γ_1 containing $\Sigma \cup \{\theta\}$ and Γ_2 containing $\Sigma \cup \{\neg\theta\}$.

2. Adding "Witnesses"

In sentential logic we could define a (unique) truth assignment from a maximal consistent set Γ and then show that it satisfied Γ. In first order logic, however, a maximal consistent set does not determine any structure – to determine a structure we need to know what the universe of the structure is and how the non-logical symbols of the language are interpreted on this set.

Given a consistent set $\Sigma \subseteq Sn_{\mathcal{L}}$, we first expand the language $\mathcal{L}$ to $\mathcal{L}'$ by adding constant symbols c_i and find a maximal consistent set $\Gamma \subseteq Sn_{\mathcal{L}'}$ which also has the property that for every formula $\psi(x)$ of $\mathcal{L}'$, $\forall x\psi(x) \in \Gamma$ iff $\psi(c_i) \in \Gamma$ for every i. We will then be able to define an $\mathcal{L}'$-structure $\mathcal{A}'$ such that every element of the universe is named by some constant sysmbol and which satisfies Γ.

To simplify the presentation we assume that the only non-logical symbol of the langauge $\mathcal{L}$ is a binary relation symbol R. We start with a consistent set $\Sigma \subseteq Sn_{\mathcal{L}}$. We define the language $\mathcal{L}' = \mathcal{L} \cup \{c_1 : i \in \mathbb{N}\}$. The consistency of Σ is not affected by this change since the set Σ says nothing about the added constants.

We next list all formulas of $\mathcal{L}'$ with just one free variable as

$$\psi_1(x), \psi_2(x), \ldots, \psi_n(x), \ldots \text{ for all } n \in \mathbb{N}.$$

(The formulas need not have the same free variable; we use x for whatever the free variable in question is). We define a sequence of sets of sentences of $\mathcal{L}'$, beginning with Σ, as follows:

let c_{i_1} be the first constant not appearing in $\psi_1(x)$, let θ_1 be $(\exists x\psi_1(x) \to \psi_1(c_{i_1})$, and let $\Sigma_1 = \Sigma \cup \{\theta_1\}$. We claim that Σ_1 is consistent. Otherwise, $\Sigma \vdash \neg\theta_1$, hence $\Sigma \vdash \exists x\psi_1(x)$ and $\Sigma \vdash \neg\psi_1(c_{i_1})$. But then, by Generalization on Constants (Theorem 9.3 in Chapter 2), we would have $\Sigma \vdash \forall x\neg\psi_1(x)$, and so Σ is inconsistent (remembering that $\exists x$ means $\neg\forall x\neg$).

We continue in this way, at the nth stage adding some sentence θ_n of the form $\exists x\psi_n(x) \to \psi_n(c_{i_n})$ to obtain a consistent set Σ_n. By Theorem 1.3 the union of all of these sets is a consistent set Σ'.

Now we apply Theorem 1.4 to obtain a maximal consistent $\Gamma \subseteq Sn_{\mathcal{L}'}$ such that $\Sigma' \subseteq \Gamma$.

We claim that Γ has the following properties:

(i) for every $\theta \in Sn_{\mathcal{L}}$, $\neg\theta \in \Gamma$ iff $\theta \notin \Gamma$,
(ii) for every $\varphi, \theta \in Sn_{\mathcal{L}}$, $(\varphi \to \theta) \in \Gamma$ iff either $\neg\varphi \in \Gamma$ or $\theta \in \Gamma$,
(iii) for every $\psi(x) \in Fm_{\mathcal{L}}$, $\forall x\psi(x) \in \Gamma$ iff $\psi(c_n) \in \Gamma$ for every $n \in \mathbb{N}$.

This claim is easily proved using the following Lemma.

Lemma 2.1. *Let Γ be maximal consistent. Then for any sentence θ, $\theta \in \Gamma$ iff $\Gamma \vdash \theta$.*

3. Defining a Structure from Γ

So, given Γ obtained as in the preceding section, we define an $\mathcal{L}'$-structure $\mathcal{A}'$ from Γ as follows:

the universe A' of the structure is $\mathbb{N}$;

${c_n}^{\mathcal{A}'} = n$ for every $n \in \mathbb{N}$;

$R^{\mathcal{A}'}(k, n)$ holds iff $R(c_k, c_n) \in \Gamma$.

We obtain the following result.

Theorem 3.1. *Let Γ and $\mathcal{A}'$ be as above Then for every sentence θ of $\mathcal{L}'$ in which $=$ does not occur, $\mathcal{A}' \models \theta$ iff $\theta \in \Gamma$.*

Proof. (outline) We prove this by induction. The base case is $R(c_k, c_n)$, which holds due to the definition of $R^{\mathcal{A}'}$. The inductive steps for the connectives are clear from parts (i) and (ii) of the Claim at the end of the preceding section. The inductive step for $\forall$ is clear from part (iii) of the Claim and the fact that $A' = \mathbb{N} = \{{c_n}^{\mathcal{A}'} : n \in \mathbb{N}\}$. $\square$

To define a structure which will also model the sentences in Γ which contain $=$ we need to allow for the possibility that $c_k = c_n \in \Gamma$ for some $k \neq n$. We choose the logical axioms for $=$ to guarantee that Γ has the following additional properties:

(iv) $c_k = c_k \in \Gamma$, if $c_k = c_n \in \Gamma$ then $c_n = c_k \in \Gamma$, and if $c_k = c_n, c_n = c_m \in \Gamma$ then $c_k = c_m \in \Gamma$;

(v) if $R(c_k, c_n)$, $c_k = c_l$, and $c_n = c_m \in \Gamma$ then $R(c_l, c_m) \in \Gamma$.

We define a structure $\mathcal{B}'$ as follows:

the universe B' of $\mathcal{B}'$ is $\{k \in \mathbb{N} : c_k \neq c_l \in \Gamma \text{ for all } l < k\}$:

${c_n}^{\mathcal{B}'}$ is the least k such that $(c_k = c_n) \in \Gamma$;

$R^{\mathcal{B}'}(k, l)$ holds iff $R(c_k, c_l) \in \Gamma$.

We then have the following, proved like Theorem 3.1 using the additional properties (iv) and (v) to check that equality statements in Γ are true in $\mathcal{B}'$.

Theorem 3.2. *For every $\theta \in Sn_{\mathcal{L}'}$. $\mathcal{B}' \models \theta$ iff $\theta \in \Gamma$.*

Since $\Sigma \subseteq \Gamma$ we have shown that the original Σ has a model, establishing Theorem 1.2.

Example. We illustrate the Henkin method with a simplified example. Let $\Sigma = \{\forall y \exists x R(y, x)\}$. Instead of listing all formulas $\psi(x)$ of $\mathcal{L}'$ we consider only the formulas $\psi_n(x)$ defined as $R(c_n, x)$. Then θ_1 is $(\exists x R(c_1, x) \rightarrow R(c_1, c_2))$, and in general θ_n is $(\exists x R(c_n, x) \rightarrow R(c_n, c_{n+1}))$, so Σ' is

$$\{\forall y \exists x (R(x, y)\} \cup \{(\exists x R(c_n, x) \rightarrow R(c_n, c_{n+1})) : n \in \mathbb{N}\}.$$

Now let $\Gamma \subseteq Sn_{\mathcal{L}'}$ be maximal consistent and contain Σ'. Then $\Gamma \vdash \exists x R(c_n, x)$ for every $n \in \mathbb{N}$ (since $(\forall y \exists x R(y, x) \rightarrow \exists x R(c_n, x))$ is an instance of Axiom 2). Therefore $\Gamma \vdash R(c_n, c_{n+1})$, and so $R(c_n, c_{n+1}) \in \Gamma$ by Lemma 2.1, for every $n \in \mathbb{N}$. Thus $\mathcal{A}' \models \forall y \exists x R(y, x)$, as desired, since $A' = \mathbb{N}$.

4. The Compactness Theorem

Just as in Sentential Logic, the Compactness Theorem is an immediate consequnce of Finiteness and the Completeness Theorem.

Theorem 4.1. (Compactness Theorem) *Let Σ be a set of sentences of $\mathcal{L}$.*

(a) *If $\Sigma \models \varphi$ then there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \varphi$.*
(b) *If every finite $\Sigma_0 \subseteq \Sigma$ has a model then Σ has a model.*

This result has many amazing applications and is of fundamental importance to the research area of Model Theory. We give two examples.

Theorem 4.2. *Let $\theta \in Sn_{\mathcal{L}}$. Assume that $\mathcal{A} \models \theta$ for every $\mathcal{A}$ with an infinite universe A. Then there is some integer n such that $\mathcal{A} \models \theta$ for every $\mathcal{A}$ whose universe A contains at least n elements.*

Proof. For every $k \in \mathbb{N}$ there is a sentence σ_k which holds of a structure iff the universe of the structure contains at least k elements. Let $\Sigma = \{\sigma_k : k \in \mathbb{N}\}$. Then $\mathcal{A} \models \Sigma$ iff the universe of $\mathcal{A}$ is infinite. Therefore the hypothesis of the Theorem implies that $\Sigma \models \theta$. By Compactness there is some finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \theta$. There must be a largest integer n such that $\sigma_n \in \Sigma_0$, and this n is then as desired. □

Theorem 4.3. *Let $\mathcal{L}$ be the language whose only non-logical symbol is a binary relation symbol $<$. Let $\mathcal{A}$ be the $\mathcal{L}$-structure $(\mathbb{N}, <)$. Then there is some $\mathcal{B} = (B, <^{\mathcal{B}})$ such that $\mathcal{B} \models \theta$ for every θ true on $\mathcal{A}$, but $\mathcal{B}$ contains "infinite" elements, that is elements b with infinitely many elements in B preceding it in the order $<^{\mathcal{B}}$.*

Proof. For every $n \in \mathbb{N}$ there is a formula $\varphi_n(x)$ of $\mathcal{L}$ which holds of an element iff there are at least n elements preceding it. Let c be a constant symbol and let $\mathcal{L}' = \mathcal{L} \cup \{c\}$. We define the set Σ of $\mathcal{L}'$-sentences as

$$\{\theta \in Sn_{\mathcal{L}} : \mathcal{A} \models \theta\} \cup \{\varphi_n(c) : n \in \mathbb{N}\}.$$

If $\mathcal{B}' \models \Sigma$ then $\mathcal{B} = (B', <^{\mathcal{B}'})$ is as desired, since $c^{\mathcal{B}'}$ is an "infinite" element of B'. Every finite $\Sigma_0 \subseteq \Sigma$ has a model $\mathcal{A}' = (\mathbb{N}, <, n_0)$ where n_0 is a sufficiently large element of $\mathbb{N}$, so Σ has a model by Compactness. □