RIGHT CLOSING ALMOST CONJUGACY FOR G-SHIFTS OF FINITE TYPE

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ABSTRACT. A G-shift of finite type (G-SFT) is a shift of finite type which commutes with the continuous action of a finite group G. For irreducible G-SFTs we classify right closing almost conjugacy, answering a question of Bill Parry.

1. INTRODUCTION

For a finite group G, a *G-shift of finite type* (G-SFT) is a shift of finite type (X, σ) together with a continuous G action on X which commutes with the shift σ. For irreducible shifts of finite type, right closing almost conjugacy is classified in terms of entropy, period, and an algebraic invariant called ideal class [6]. Bill Parry [15] posed the following question: what additional invariants are necessary to classify right closing almost conjugacy for irreducible G-SFTs? With Theorem 4.1 we show that for mixing G-SFTs where the G action is free, there are no additional invariants. In section 5 we generalize Theorem 4.1 to mixing G-SFTs where the G action is no longer assumed to be free. In section 6 we generalize further to irreducible but periodic G-SFTs. As a corollary to our results we classify regular isomorphism for G-Markov chains with respect to measures of maximal entropy.

Without the right closing assumption, almost conjugacy for irreducible G-SFTs was classified by Roy Adler, Bruce Kitchens and Brian Marcus [2]. They were working in a more general setting, but by modifying the proofs given here we can arrive at the same classification of almost conjugacy for irreducible G-SFTs (as was also done in [14]).

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2. BACKGROUND AND DEFINITIONS

We assume some familiarity with shifts of finite type; [11] and [12] provide more complete backgrounds. All of the free G-SFTs we consider arise out of skew products, as in [7]. The study of skew products dates back to von

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Neumann, in the context of ergodic measure preserving transformations on a probability space. For an example of more recent work with skew products in ergodic theory, see [10]. Also see [16] and [17] (and their references) for recent results with skew products in Livsic theory.

2.1. **Shifts of finite type.** Let A be an $n \times n$ matrix over the nonnegative integers $\mathbb{Z}_+$. Then A is the adjacency matrix for a directed graph, $\mathcal{G}_A$, which has vertices $\{v_1, v_2, \ldots, v_n\}$, and the number of edges from v_I to v_J is A_{IJ}. Let $\mathcal{E}_A = \{\text{edges in } \mathcal{G}_A\}$, and put

$$\Sigma_A = \{x = (x_i)_{i\in\mathbb{Z}} \in (\mathcal{E}_A)^{\mathbb{Z}} \,|\, \text{each } x_i x_{i+1} \text{ is a path in } \mathcal{G}_A\}.$$

With the appropriate topology (the relative of the product of the discrete topology on $\mathcal{E}_A$), Σ_A is a compact metric space. The *shift* on Σ_A is the homeomorphism $\sigma : \Sigma_A \to \Sigma_A$ given by $(\sigma x)_i = x_{i+1}$. The pair (Σ_A, σ) is the *edge shift of finite type (SFT)* defined by A. Where σ is understood, we write just Σ_A to denote (Σ_A, σ).

A *map* between SFTs $\pi\colon \Sigma_A \to \Sigma_B$ is a continuous function such that $\pi \circ \sigma(x) = \sigma \circ \pi(x)$ for all $x \in \Sigma_A$. The map π is *one block* if it is induced by a function which sends each edge of $\mathcal{G}_A$ to an edge of $\mathcal{G}_B$. A *factor map* is a surjective map. An injective factor map is a *conjugacy.*

The matrix A is *irreducible* if for each entry A_{IJ} of A there is a natural number N such that $(A^N)_{IJ} > 0$. If A is irreducible we say also that the graph $\mathcal{G}_A$ and the edge SFT Σ_A are irreducible. The matrix A is *primitive* if there is a natural number N such that for each entry A_{IJ} of A, $(A^N)_{IJ} > 0$. If A is primitive we say also that the graph $\mathcal{G}_A$ is primitive; in this case the edge SFT Σ_A is *mixing.*

If λ is the Perron eigenvalue of A, then the *entropy* of Σ_A is $\log \lambda$. If $v = [v_1, v_2, \ldots, v_n]^T$ is a right Perron eigenvector with entries in the ring $\mathbb{Z}[1/\lambda]$, then the *ideal class* of Σ_A is the class of the $\mathbb{Z}[1/\lambda]$-ideal which is generated by the components $v_1, \ldots, v_n$ of v. A point $x \in \Sigma_A$ is *periodic* if there exists a natural number p such that $\sigma^p(x) = x$. In this case p is a *period* of x; the smallest period of x is called the *least period* of x. We define the period of the edge SFT Σ_A to be the greatest common divisor of the set of periods of periodic points in Σ_A. The period of the graph $\mathcal{G}_A$ is the period of Σ_A.

Any SFT (X, σ) is conjugate to some edge SFT (Σ_A, σ). Then, the terms *irreducible*, *mixing*, *entropy*, *ideal class* and *period* apply to X exactly as they apply to Σ_A. A point $x \in X$ is *doubly transitive* if both sets $\{\sigma^n(x)\colon n \geq 0\}$ and $\{\sigma^n\colon n \leq 0\}$ are dense in X. Two points $x = (x_n)_{n\in\mathbb{Z}}$ and $y = (y_n)_{n\in\mathbb{Z}}$ in X are *left asymptotic* if there is an integer n such that $x_k = y_k$ for all $k \leq n$. A map between SFTs $\pi\colon X \to Y$ is 1-1 *a.e.* if it is injective on the set of doubly transitive points in X. The map π is *right closing* if, for each pair $x, y \in X$ of distinct left asymptotic points, $\pi(x) \neq \pi(y)$. We say the SFTs X and Y are *right closing almost conjugate as SFTs* if there is a third

SFT Z which factors onto both X and Y by factor maps which are 1-1 *a.e.* and right closing.

2.2. **G-shifts of finite type.** Let G be a finite group. A *G-SFT* is an SFT (X, σ) together with a continuous right G action on X such that $\sigma(x \cdot g) = \sigma(x) \cdot g$ for all $x \in X$ and $g \in G$. We say the G-SFT X (or the G action on X) is *free* if, for each non-identity element g of G, $x \cdot g \neq x$ for all $x \in X$. We say X (or the G action on X) is *faithful* if, for each non-identity element g of G, there exists some $x \in X$ such that $x \cdot g \neq x$. If Y is another G-SFT, then a *G-map* $\pi \colon X \to Y$ is a map between SFTs such that $\pi(x \cdot g) = \pi(x) \cdot g$ for all $x \in X$ and $g \in G$. A *G-factor map* is a surjective G-map and a *G-conjugacy* is an injective G-factor map. Two G-SFTs X and Y are *right closing almost conjugate as G-SFTs* if there is a third G-SFT Z which factors onto both X and Y by 1-1 *a.e.* and right closing G-factor maps. We point out that right closing almost conjugate G-SFTs are in particular right closing almost conjugate SFTs. The terms we define above for SFTs, such as *irreducible*, *mixing*, *entropy*, *ideal class* and *period*, apply to a G-SFT X as they apply to X as an SFT.

2.3. **Skew products and matrices over $\mathbb{Z}_+G$.** By $\mathbb{Z}G$ we mean the integral group ring of G. We write an element x of $\mathbb{Z}G$ as $x = \sum_{g \in G} n_g g$, where each $n_g \in \mathbb{Z}$. Then for each g in G we define $\pi_g(x) = n_g$. If $\pi_g(x) > 0$, then g is a *summand* of x. If $\pi_g(x) > 0$ for each g in G, then we say x is *very positive* and write $x \gg 0$. The *augmentation* of x is $|x| = \sum_{g \in G} \pi_g(x)$. If A is a matrix over $\mathbb{Z}G$, then $A \gg 0$ if $A_{IJ} \gg 0$ for each entry A_{IJ} of A. The augmentation $|A|$ is the matrix given by $|A|_{IJ} = |A_{IJ}|$ for each entry A_{IJ} of A. We let $\mathbb{Z}_+G = \{x \in \mathbb{Z}G \colon \pi_g(x) \geq 0 \text{ for each } g \in G\}$.

If $\mathcal{G}$ is a directed graph and l is a labeling of the edges of $\mathcal{G}$ by elements of G, then we say $(\mathcal{G}, l)$ is a *G-labeled graph.* If A is a square matrix over $\mathbb{Z}_+G$, then $|A|$ is a square matrix over $\mathbb{Z}_+$ which, as before, is the adjacency matrix for a directed graph $\mathcal{G}_{|A|}$. The matrix A corresponds to a G-labeled graph $(\mathcal{G}_{|A|}, l_A)$, where l_A is defined as follows: for each pair I, J of vertices in $\mathcal{G}_{|A|}$, $A_{IJ} = \sum n_g g$ if and only if for each $g \in G$ exactly n_g of the edges from I to J are l_A-labeled g. The edge labeling l_A determines a function $\tau_A \colon \Sigma_{|A|} \to G$ by $\tau_A(x) = l_A(x_0)$ for each $x = (x_n)_{n \in \mathbb{Z}}$ in $\Sigma_{|A|}$. The function τ_A is *locally constant*: for each $x \in \Sigma_{|A|}$, τ_A is constant on a neighborhood of x (here τ_A is constant on $\{y \in \Sigma_{|A|} \mid y_0 = x_0\}$). The function τ_A is the *skewing function* defined by A. Given two locally constant functions $\tau_1, \tau_2 \colon \Sigma_{|A|} \to G$, we say τ_1 is *cohomologous* to τ_2 if there is another locally constant $h \colon \Sigma_{|A|} \to G$ such that $\tau_1(x) = [h(\sigma x)]^{-1} \cdot \tau_2(x) \cdot h(x)$ for each $x \in \Sigma_{|A|}$.

The $\mathbb{Z}_+G$ matrix A determines an automorphism $S_A \colon \Sigma_{|A|} \times G \to \Sigma_{|A|} \times G$ by $S_A(x, g) = (\sigma(x), \tau_A(x) \cdot g)$, where τ_A is the skewing function defined by A. We say the dynamical system $(\Sigma_{|A|} \times G, S_A)$ is the *skew product* defined by A. There is a free right G action on $(\Sigma_{|A|} \times G, S_A)$ which commutes with the automorphism S_A, given by $g\colon (x,h) \mapsto (x, h \cdot g)$. Often we write just S_A as an abbreviation for the skew product $(\Sigma_{|A|} \times G, S_A)$

We can present the skew product S_A as a free G-SFT (which we also denote by S_A) as follows. As an edge SFT S_A has graph $\mathcal{G}$, where the vertex set of $\mathcal{G}$ is the product of the vertex set of $\mathcal{G}_{|A|}$ with G, and for each edge e from I to J in $\mathcal{G}_{|A|}$, for each g in G, there is an edge from (I,g) to $(J, l_A(e)\cdot g)$ in $\mathcal{G}$. For each pair of vertices v, v' of $\mathcal{G}$ we choose an ordering of the edges from v to v', and let g in G act by the one block map given by the unique automorphism of $\mathcal{G}$ which acts on the vertex set of $\mathcal{G}$ by $(J,h) \mapsto (J, h \cdot g)$, and which is order preserving.

In this way any skew product is a free G-SFT. Conversely, any free G-SFT is G-conjugate to a skew product S_A for some $\mathbb{Z}_+G$ matrix A. We say a matrix A over $\mathbb{Z}_+G$ is *very primitive* if there exists a natural number N such that $A^N \gg 0$. One easily checks that A is very primitive if and only if the G-SFT S_A is mixing.

Square matrices A and B over $\mathbb{Z}_+G$ are *strong shift equivalent* (SSE) over $\mathbb{Z}_+G$ if they are connected by a string of elementary moves of the following sort: there are R and S over $\mathbb{Z}_+G$ such that $A = RS$ and $B = SR$. Parry has shown that A and B are SSE over $\mathbb{Z}_+G$ if and only if the skew products S_A and S_B are G-conjugate [7, Prop. 2.7.1].

3. Some Useful Results

In this section we collect some results to be used later. We begin with the known classification of right closing almost conjugacy for irreducible SFTs, which is a corollary of [6, Theorem 7.1].

Theorem 3.1. *Irreducible SFTs are right closing almost conjugate as SFTs if and only if they have the same ideal class, entropy and period.*

Lemma 3.2 ($\mathbb{Z}_+G$ **Masking Lemma).** *Let A and C be matrices over $\mathbb{Z}_+G$ such that the skew product S_A is G-conjugate to a subsystem of the skew product S_C. Then there is a matrix B over $\mathbb{Z}_+G$ such that A is a principal submatrix of B, and S_B and S_C are G-conjugate skew products.*

Proof. If S_A is G-conjugate to a subsystem of S_C, then A is SSE over $\mathbb{Z}_+G$ to a principal submatrix of C [7, Prop. 2.7.1]. Nasu's original Masking Lemma for matrices over $\mathbb{Z}$ [13, Lemma 3.18] also holds for matrices over an arbitrary semiring containing 0 and 1 [5, Appendix 1]; in particular it holds for matrices over $\mathbb{Z}_+G$. This means there is a matrix B over $\mathbb{Z}_+G$ such that A is a principal submatrix of B, and B is SSE over $\mathbb{Z}_+G$ to C; S_B and S_C are G-conjugate skew products by [7, Prop. 2.7.1].

□

Lemma 3.3. *Let A and B be matrices over $\mathbb{Z}_+G$. A G-factor map $\pi\colon S_A \to S_B$ induces a factor map $\overline{\pi}\colon \Sigma_{|A|} \to \Sigma_{|B|}$ such that the skewing function τ_A is cohomologous to $\tau_B \circ \overline{\pi}$. Conversely, if $\overline{\pi}\colon \Sigma_{|A|} \to \Sigma_{|B|}$ is a factor map such that τ_A is cohomologous to $\tau_B \circ \overline{\pi}$, then $\overline{\pi}$ induces a G-factor map*

$\pi\colon S_A \to S_B$. The G-map π is 1-1 *a.e. and right closing if and only if the map $\overline{\pi}$ is* 1-1 *a.e. and right closing.*

Proof. Let $\pi\colon S_A \to S_B$ be a G-factor map. Write $\pi = \pi_1 \times \pi_2$, so that for an element $(x,g) \in \Sigma_{|A|} \times G$, $\pi(x,g) = (\pi_1(x,g), \pi_2(x,g))$. Let e denote the identity element of G. Then $\pi\colon (x,g) \mapsto (\pi_1(x,e), \pi_2(x,e) \cdot g)$, since π intertwines G actions. For $x \in \Sigma_{|A|}$, set $\overline{\pi}(x) = \pi_1(x,e)$ and $h(x) = \pi_2(x,e)$, so that $\pi(x,g) = (\overline{\pi}(x), h(x)\cdot g)$. Look componentwise at the equality $\pi \circ S_A = S_B \circ \pi$. The first component shows that $\overline{\pi}\colon \Sigma_{|A|} \to \Sigma_{|B|}$ is a well-defined factor map. The second component shows that $\tau_A(x) = [h(\sigma x)]^{-1} \cdot (\tau_B \circ \overline{\pi})(x) \cdot h(x)$ for each $x \in \Sigma_{|A|}$. Hence τ_A is cohomologous to $\tau_B \circ \overline{\pi}$.

Conversely, suppose $\overline{\pi}\colon \Sigma_{|A|} \to \Sigma_{|B|}$ is a factor map such that τ_A is cohomologous to $\tau_B \circ \overline{\pi}$. Then there is a locally constant map $h\colon \Sigma_{|A|} \to G$ such that for each $x \in \Sigma_{|A|}$, $\tau_A(x) = [h(\sigma x)]^{-1} \cdot (\tau_B \circ \overline{\pi})(x) \cdot h(x)$. Define $\pi\colon \Sigma_{|A|} \times G \to \Sigma_{|B|} \times G$ by $\pi(x,g) = (\overline{\pi}(x), h(x) \cdot g)$. Observe that π is a G-factor map.

For the last statement of the lemma, consider the following commutative diagram, where the maps $q_A\colon S_A \to \Sigma_A$ and $q_B\colon S_B \to \Sigma_B$ are each given by $(x,g) \mapsto x$.

$$\begin{CD} S_A @>{\pi}>> S_B \\ @V{q_A}VV @VV{q_B}V \\ \Sigma_{|A|} @>{\overline{\pi}}>> \Sigma_{|B|} \end{CD}$$

Both maps q_A and q_B are $|G|$-to-1 everywhere. Therefore π is 1-1 *a.e.* if and only if $\overline{\pi}$ is 1-1 *a.e.* For the closing condition, note that if ϕ and ψ are maps between irreducible SFTs, then $\phi \circ \psi$ is right closing if and only if both ϕ and ψ are right closing [6, Props. 4.10 and 4.11]. Because the constant-to-one maps q_A and q_B are in particular right closing [11, Prop. 4.3.4], it follows that π is right closing if and only if $\overline{\pi}$ is right closing.

□

If $(\mathcal{G}, l)$ is a G-labeled graph, then for a cycle $s = s_1 s_2 \dots s_p$ in $\mathcal{G}$ we define the *weight* of s by $l(s) = l(s_1)l(s_2)\cdots l(s_p)$. The *ratio group* Δ_l is the subgroup of G given by

$$\Delta_l = \{l(s) \cdot l(s')^{-1} \colon s, s' \text{ are cycles in } \mathcal{G} \text{ of the same length}\}$$

Theorem 3.4 (**$\mathbb{Z}G$ Replacement Theorem**)**.** *Let $(\mathcal{G}, l)$ and $(\mathcal{G}', l')$ be irreducible G-labeled graphs of the same period which define edge SFTs Σ and Σ' (respectively) and skewing functions $\tau\colon \Sigma \to G$ and $\tau'\colon \Sigma' \to G$ given by $\tau(x) = l(x_0)$ and $\tau'(x) = l'(x_0)$. Let $\pi\colon \Sigma \to \Sigma'$ be a factor map such that τ is cohomologous to $\tau' \circ \pi$. If $\Delta_l = \Delta_{l'}$, then there is a* 1-1 *a.e.*

factor map $\overline{\pi}\colon \Sigma \to \Sigma'$ *such that* τ *is cohomologous to* $\tau' \circ \overline{\pi}$*. Moreover, if* π *is right closing, then* $\overline{\pi}$ *can be taken to be right closing as well.*

In [1, Theorem 6.1], Ashley proves a version of his ($\mathbb{Z}$) Replacement Theorem for maps between irreducible Markov chains, which can be interpreted as follows. Let $\mathbb{R}^+$ denote the multiplicative group of positive real numbers. Let $(\mathcal{G}, l)$ and $(\mathcal{G}', l')$ be irreducible $\mathbb{R}^+$-labeled graphs of the same period, which define irreducible SFTs Σ and Σ' and locally constant functions $\tau\colon \Sigma \to \mathbb{R}^+$ and $\tau'\colon \Sigma' \to \mathbb{R}^+$ where $\tau(x) = l(x_0)$ and $\tau'(x) = l'(x_0)$. If the ratio groups Δ_l and $\Delta_{l'}$ are equal (as multiplicative subgroups of $\mathbb{R}^+$), and $\pi\colon \Sigma \to \Sigma'$ is a factor map such that τ is cohomologous to $\tau' \circ \pi$, then there is a 1-1 *a.e.* factor map $\overline{\pi}\colon \Sigma \to \Sigma'$ such that τ is cohomologous to $\tau' \circ \overline{\pi}$. Moreover, if π is right closing, then $\overline{\pi}$ can be taken to be right closing as well.

If instead of $\mathbb{R}^+$-labeled graphs we consider G-labeled graphs, then we have the statement of Theorem 3.4. To prove Theorem 3.4, one can easily check that Ashley's proof for $\mathbb{R}^+$-labeled graphs goes through for G-labeled graphs as well.

Theorem 3.5. *Let* X *and* Y *be mixing free* G*-SFTs. Let* $\pi\colon X \to Y$ *be a* G*-factor map which is right closing. Then there is a* G*-factor map* $\pi'\colon X \to Y$ *which is* 1-1 *a.e. and right closing.*

Proof. Since X and Y are mixing free G-SFTs, assume without loss of generality that $X = S_A$ and $Y = S_B$ for very primitive matrices A and B over $\mathbb{Z}_+G$. By Lemma 3.3 the G-factor map π induces a map $\overline{\pi}\colon \Sigma_{|A|} \to \Sigma_{|B|}$ such that τ_A is cohomologous to $\tau_B \circ \overline{\pi}$. Since A and B are very primitive the periods of $\mathcal{G}_{|A|}$ and $\mathcal{G}_{|B|}$ are both 1, and furthermore $\Delta_{l_A} = \Delta_{l_B} = G$. So assume (by Theorem 3.4) that the map $\overline{\pi}$ is 1-1 *a.e.* and right closing. Again apply Lemma 3.3 to obtain a G-factor map $\pi'\colon S_A \to S_B$ which is 1-1 *a.e.* and right closing. □

4. Right closing almost conjugacy for mixing free G-SFTs

For mixing SFTs, entropy and ideal class are a complete set of invariants of right closing almost conjugacy (Theorem 3.1). We show that there are no additional invariants of right closing almost conjugacy for mixing free G-SFTs.

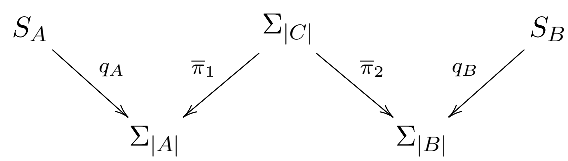
Theorem 4.1. *Let* X *and* Y *be mixing free* G*-SFTs. Then the following are equivalent.*

(1) X *and* Y *are right closing almost conjugate as* G*-SFTs.*
(2) X *and* Y *are right closing almost conjugate as SFTs.*
(3) X *and* Y *have the same entropy and ideal class.*

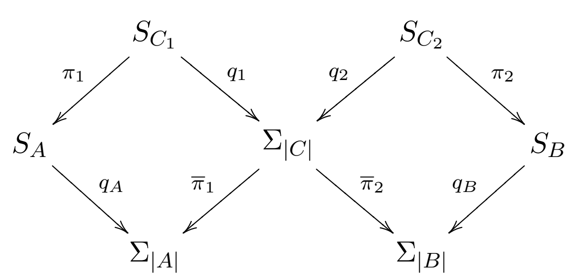
Moreover, assuming (2) *or* (3)*, the common extension of* X *and* Y *in* (1) *can be taken to be a free* G*-SFT.*

Proof. (2) $\Leftrightarrow$ (3) follows from Theorem 3.1. Right closing almost conjugate G-SFTs are in particular right closing almost conjugate as SFTs, so (1) $\Rightarrow$ (2). It remains to show (2) $\Rightarrow$ (1).

Let X and Y be mixing free G-SFTs which are right closing almost conjugate as SFTs. Without loss of generality, assume that X and Y are skew products S_A and S_B for very primitive matrices A and B over $\mathbb{Z}_+G$. Let l_A, l_B, τ_A and τ_B denote the edge labelings and skewing functions defined by A and B, respectively (see section 2). Since S_A and S_B are right closing almost conjugate as SFTs, they have the same entropy and ideal class (Theorem 3.1). The factor maps $q_A \colon S_A \to \Sigma_{|A|}$ and $q_B \colon S_B \to \Sigma_{|B|}$ given by $(x, g) \mapsto x$ are $|G|$-to-1 everywhere. In particular they preserve entropy and ideal class, so $\Sigma_{|A|}$ and $\Sigma_{|B|}$ have the same entropy and ideal class. Hence $\Sigma_{|A|}$ and $\Sigma_{|B|}$ are right closing almost conjugate as SFTs (Theorem 3.1). Let $\Sigma_{|C|}$ be a common extension of $\Sigma_{|A|}$ and $\Sigma_{|B|}$ by 1-1 *a.e.* right closing factor maps $\overline{\pi}_1 \colon \Sigma_{|C|} \to \Sigma_{|A|}$ and $\overline{\pi}_2 \colon \Sigma_{|C|} \to \Sigma_{|A|}$.



Without loss of generality, assume the factor maps $\overline{\pi}_1$ and $\overline{\pi}_2$ are one block. Define edge labelings l_1 and l_2 on $\mathcal{G}_{|C|}$ by $l_1 = l_A \circ \overline{\pi}_1$ and $l_2 = l_B \circ \overline{\pi}_2$. The labelings l_1 and l_2 correspond to matrices C_1 and C_2 (respectively) over $\mathbb{Z}_+G$ such that $|C_1| = |C_2| = |C|$. Define skewing functions $\tau_1 \colon \Sigma_{|C|} \to G$ and $\tau_2 \colon \Sigma_{|C|} \to G$ by $\tau_1(x) = l_1(x_0)$ and $\tau_2(x) = l_2(x_0)$. Define G-factor maps $\pi_1 \colon S_{C_1} \to S_A$ and $\pi_2 \colon S_{C_2} \to S_B$ by $\pi_1(x, g) = (\overline{\pi}_1(x), g)$ and $\pi_2(x, g) = (\overline{\pi}_2(x), g)$. Let $q_1 \colon S_{C_1} \to \Sigma_{|C|}$ and $q_2 \colon S_{C_2} \to \Sigma_{|C|}$ be the factor maps $(x, g) \mapsto x$. Then the following diagram commutes.



The factor maps $\overline{\pi}_1$ and $\overline{\pi}_2$ are 1-1 *a.e.* and right closing, so the factor maps π_1 and π_2 are as well (Lemma 3.3). In particular S_{C_1} and S_{C_2} are mixing free G-SFTs, so C_1 and C_2 are very primitive. Let l be the $(G \times G)$-labeling $l = l_1 \times l_2$. Then l corresponds to a $\mathbb{Z}_+(G \times G)$ matrix whose augmentation is $|C|$. Call this matrix C. Let $\tau\colon \Sigma_{|C|} \to G \times G$ denote the skewing function given by $\tau(x) = l(x_0)$.

Claim 4.2. *There is a vertex I in $\mathcal{G}_{|C|}$ and a natural number N such that there is a collection $\mathcal{U}$ of paths of length N from I to I with the following properties:*

(1) *for each g in G there are at least $|G|$ paths $u \in \mathcal{U}$ with weights $l_1(u) = g$.*
(2) *for each g in G there are at least $|G|$ paths $u \in \mathcal{U}$ with weights $l_2(u) = g$.*
(3) *for each $u = u_1 u_2 \cdots u_N \in \mathcal{U}$ the point $x^u \in \Sigma_{|C|}$, defined by $x^u_i = u_j$ if $i \equiv j \mod N$, has least period N.*
(4) *if u and v are distinct paths in $\mathcal{U}$, then x^u and x^v are in different orbits under the shift.*

To prove the claim, let α be the element of $\mathbb{Z}_+ G$ given by $\alpha = \sum_{g\in G} g$. Fix a vertex I in $\mathcal{G}_{|C|}$. Let η be the number of cycles of length 1 in $\mathcal{G}_{|C|}$, and choose a positive integer k large enough so that $k - \eta \geq |G|$. Since C_1 and C_2 are very primitive matrices there is a positive integer $M = M(k)$ such that, for $i = 1, 2$ and for all $m \geq M$, $k \cdot \alpha$ is a summand of $(C_i^m)_{II}$. Let $N \geq M$ be a prime number. Let $\mathcal{V}$ be the set of all N-paths from I to I. Each $v = v_1 v_2 \cdots v_N \in \mathcal{V}$ defines a point $x^v \in \Sigma_{|C|}$ by $x^v_i = v_j$ if $i \equiv j \mod N$. Since N is prime, each such x^v has least period either N or 1. Let $\mathcal{V}^1 = \{v \in V : x^v \text{ has least period } 1\}$ and $\mathcal{U} = \mathcal{V} - \mathcal{V}^1$.

It remains to verify that $\mathcal{U}$ satisfies the properties of the claim. Note that, for $i = 1, 2$, each monomial summand g of $(C_i^N)_{II}$ corresponds to a path $v \in \mathcal{V}$ with weight $l_i(v) = g$. Also, N was chosen so that $k \cdot \alpha$ is a summand of each $(C_i^N)_{II}$. So for $i = 1, 2$ and for each $g \in G$, there are at least k paths $v \in \mathcal{V}$ with weight $l_i(v) = g$. There are only η cycles of length 1 in $\mathcal{G}_{|C|}$, so in particular $|\mathcal{V}^1| \leq \eta$. But $k - \eta \geq |G|$. Hence, for $i = 1, 2$ and for each $g \in G$, there are at least k paths $u \in \mathcal{U}$ with weight $l_i(u) = g$, which verifies properties (1) and (2). Properties (3) and (4) are true by construction of $\mathcal{U}$. This proves the claim.

Now consider all points $x^u \in \Sigma_{|C|}$ such that $u \in \mathcal{U}$. Let $\overline{\Sigma}_{|C|}$ denote the smallest closed σ-invariant subset of $\Sigma_{|C|}$ containing all points of this form. Then $\overline{\Sigma}_{|C|} \times G$ is a closed S_C-invariant subset of $\Sigma_{|C|} \times G$, so it is a subsystem of the skew product S_C. Let $\overline{S}_C$ denote this subsystem of S_C.

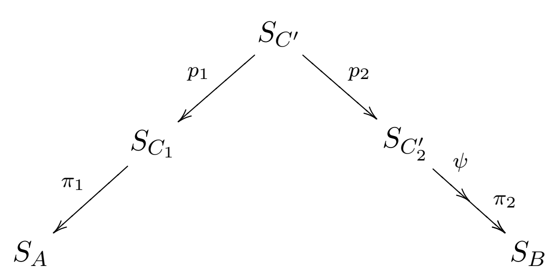
Construct a $(G \times G)$-labeled graph $(\mathcal{H}, l_{\mathcal{H}})$ as follows. The vertex set of $\mathcal{H}$ consists of N vertices, $I_1, I_2, \ldots, I_N$. For $j = 1, 2, \ldots, N-1$, draw exactly one edge starting at I_j and ending at I_{j+1}, and give this edge the $l_{\mathcal{H}}$-label (e, e), where e is the identity element of G. From I_N to I_1 draw exactly $|\mathcal{U}|$ edges, call them $s_1, s_2, \ldots, s_{|\mathcal{U}|}$. Let $\mathcal{S} = \{s_1, s_2, \ldots, s_{|\mathcal{U}|}\}$, and fix a set bijection $\phi\colon \mathcal{S} \to \mathcal{U}$. For $s_i \in \mathcal{S}$, put $l_{\mathcal{H}}(s_i) = l(\phi(s_i)) = (l_1(\phi(s_i)), l_2(\phi(s_i)))$.

Let D be the $\mathbb{Z}_+(G\times G)$ adjacency matrix for the $(G\times G)$-labeled graph $(\mathcal{H}, l_{\mathcal{H}})$. Observe that the set bijection $\phi\colon \mathcal{S} \to \mathcal{U}$ induces a $(G\times G)$-conjugacy between S_D and $\overline{S}_C$. Assume without loss of generality that D is a principal submatrix of C (Lemma 3.2), so that $(\mathcal{H}, l_{\mathcal{H}})$ is an induced sub-labeled graph of $(\mathcal{G}_{|C|}, l)$.

For each $g \in G$, at least $|G|$ of the edges $s_i \in \mathcal{S}$ have l-labels of the form $(g, \cdot)$, and at least $|G|$ of the $s_i \in \mathcal{S}$ have l-labels of the form $(\cdot, g)$ (by definition). Therefore there is a way to permute the second coordinates of the l-labelings of edges in $\mathcal{S}$ such that each $(g,h) \in G\times G$ labels at least one $s_i \in \mathcal{S}$. Equivalently, there exists a graph isomorphism $\overline{P}$ of $\mathcal{G}_{|C|}$ which fixes all edges except those in $\mathcal{S}$, and permutes the set $\mathcal{S}$ such that for any $(g,h) \in G\times G$, there is at least one edge $s_i \in \mathcal{S}$ with $(l_1(s_i), l_2\circ\overline{P}(s_i)) = (g,h)$. Fix a graph isomorphism $\overline{P}$ with this property and set l' to be the $(G\times G)$-labeling of $\mathcal{G}_{|C|}$ given by $l' = l_1 \times (l_2 \circ \overline{P})$. Let P denote the automorphism of $\Sigma_{|C|}$ induced by $\overline{P}$. Let C_2' be the $\mathbb{Z}_+G$ matrix defined by the edge labeling $l_2\circ\overline{P}$ of $\mathcal{G}_{|C|}$. Note that the map $\psi\colon S_{C_2'} \to S_{C_2}$ given by $(x,g) \mapsto (P(x), g)$ is a G-conjugacy.

Let C' be the $\mathbb{Z}_+(G\times G)$ matrix defined by the edge labeling l' of $\mathcal{G}_{|C|}$, and let $\tau'\colon \Sigma_{|C|} \to G\times G$ be the skewing function given by $\tau'(x) = l'(x_0)$. Then $S_{C'}$ is the skew product $(\Sigma_{|C|}\times G\times G, S_{C'})$, where $S_{C'}(x,g,h) = (\sigma(x), \tau'(g,h)) = (\sigma(x), \tau_1(x)\cdot g, (\tau_2\circ P)(x)\cdot h)$, and $G\times G$ acts by $(k,l)\colon (x,g,h) \mapsto (x, gk, hl)$. Note that C' is very primitive. (This is because, with $I = I_1$ and N as above, $(C'^N)_{II}$ has as a summand every element of $G\times G$.) Therefore $S_{C'}$ is a mixing free $(G\times G)$-SFT.

From now on, regard $S_{C'}$ as a mixing free G-SFT by restricting the $(G\times G)$-action to the diagonal: let an element $g\in G$ act by $(x,h,k)\mapsto (x, hg, kg)$. Let $p_1\colon S_{C'} \to S_{C_1}$ be the $|G|$-to-one factor map $(x,g,h)\mapsto (x,g)$, and let $p_2\colon S_{C'} \to S_{C_2'}$ be the $|G|$-to-one factor map $(x,g,h)\mapsto(x,h)$. Note that p_1 and p_2 are G-factor maps; they are right closing because they are constant-to-one [11, Prop 4.3.4]. This gives a diagram of right closing G-factor maps:



$S_{C'}$ is a mixing free G-SFT, so by Theorem 3.5, the right closing G-factor maps $\pi_1\circ p_1$ and $\pi_2\circ\psi\circ p_2$ can be replaced by 1-1 *a.e.* and right closing G-factor maps. This proves the theorem.

$\square$

5. GENERAL MIXING G-SFTS

In this section we classify right closing almost conjugacy for mixing G-SFTs where the G action is no longer assumed to be free. We will need this generalization to classify the irreducible but periodic case in section 6. We begin with a result for faithful G-SFTs, which were defined in section 2.

Lemma 5.1. *Any irreducible faithful G-SFT is a 1-1 a.e. right closing G-factor of an irreducible free G-SFT.*

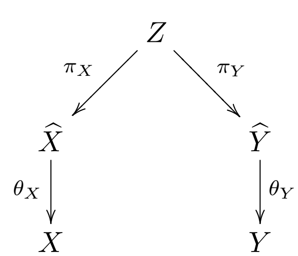
Lemma 5.1 is a corollary of [2, Theorem 3]. If X is a G-SFT, we let H^X denote the normal subgroup of G which acts by the identity map. Then X is a faithful (G/H^X)-SFT where, for all $g \in G$ and $x \in X$, $x \cdot (gH^X) = x \cdot g$.

Theorem 5.2. *Let X and Y be mixing G-SFTs. Then the following are equivalent.*

1. *X and Y are right closing almost conjugate as G-SFTs*
2. *X and Y are right closing almost conjugate as SFTs, and $H^X = H^Y$.*
3. *X and Y have the same entropy and ideal class, and $H^X = H^Y$.*

Proof. (2) $\Leftrightarrow$ (3) follows from Theorem 3.1. If X and Y are right closing almost conjugate as G-SFTs, then in particular they are right closing almost conjugate as SFTs. Moreover, if Z is a common 1-1 *a.e.* right closing G-extension of X and Y, then $H^X = H^Z$ and $H^Y = H^Z$, because 1-1 *a.e.* G-factor maps preserve the subgroup H^Z. This proves (1) $\Rightarrow$ (2).

Conversely, suppose X and Y are right closing almost conjugate as SFTs, and $H = H^X = H^Y$. Then X and Y are faithful (G/H)-SFTs, where for all $x \in X$, $y \in Y$ and $g \in G$, $x \cdot (gH) = x \cdot g$ and $y \cdot (gH) = y \cdot g$. Hence there are free (G/H)-SFTs $\widehat{X}$ and $\widehat{Y}$, and 1-1 *a.e.* right closing (G/H)-factor maps $\theta_X : \widehat{X} \to X$ and $\theta_Y : \widehat{Y} \to Y$ (Lemma 5.1). Since X and Y are right closing almost conjugate as SFTs, they have the same entropy and ideal class. Since θ_X and θ_Y are right closing factor maps between irreducible SFTs, they preserve entropy and ideal classes. So $\widehat{X}$ and $\widehat{Y}$ have the same entropy and ideal class, and are therefore right closing almost conjugate as SFTs. Thus $\widehat{X}$ and $\widehat{Y}$ are right closing almost conjugate as (G/H)-SFTs, and the common extension can be taken to be a free (G/H)-SFT (Theorem 4.1). Let Z be a free (G/H)-SFT with 1-1 *a.e.* right closing (G/H)-factor maps $\pi_X : Z \to \widehat{X}$ and $\pi_Y : Z \to \widehat{Y}$.



For all $\widehat{x} \in \widehat{X}$, $\widehat{y} \in \widehat{Y}$ and $g \in G$, put $\widehat{x} \cdot g = \widehat{x} \cdot (gH)$ and $\widehat{y} \cdot g = \widehat{y} \cdot (gH)$. With these G actions, $\widehat{X}$ and $\widehat{Y}$ are G-SFTs, and θ_X and θ_Y are now G-maps. For all $z \in Z$ and $g \in G$, put $g \cdot z = z \cdot (gH)$. This G action makes Z a G-SFT as well, and π_X and π_Y are now G-maps. Thus Z together with the maps $\theta_X \circ \pi_X$ and $\theta_Y \circ \pi_Y$ give a right closing almost conjugacy between X and Y as G-SFTs.

□

6. The irreducible but periodic case

Here we classify right closing almost conjugacy for irreducible but periodic G-SFTs. If (X, σ) is an irreducible G-SFT of period p, then we let $X^0, X^1, \dots, X^{p-1}$ denote the cyclically moving subsets of X under σ. Then for $0 \le n \le p-1$, (X^n, σ^p) is a mixing SFT. The (X^n, σ^p) are pairwise conjugate SFTs and the action of G on (X, σ) permutes the (X^n, σ^p). If the entropy of (X, σ) is $\log(\lambda)$, then the entropy of each (X^n, σ^p) is $\log(\lambda^p)$. The ideal class (in $\mathbb{Z}[1/\lambda^p]$) of (X^n, σ^p) is determined by the ideal class (in $\mathbb{Z}[1/\lambda]$) of (X, σ). We let $\overline{X} = X^0$ and $\overline{\sigma} = \sigma^p|_{\overline{X}}$. Then as SFTs, X is conjugate to $\overline{X} \times \{0, \dots, p-1\}$, where the shift for the latter is given by

$$\sigma(\overline{x}, n) = \begin{cases} (\overline{x}, n+1) & \text{if } 0 \le n \le p-2, \\ (\overline{\sigma}(\overline{x}), 0) & \text{if } n = p-1. \end{cases} \tag{6.1}$$

We give to $\overline{X} \times \{0, \dots, p-1\}$ the G action which is the image under conjugacy of the G action on X, so that X is G-conjugate to $\overline{X} \times \{0, \dots, p-1\}$. Without loss of generality, we assume from now on that irreducible but periodic G-SFTs are of the form $(X, \sigma) = (\overline{X} \times \{0, \dots, p-1\}, \sigma)$, where the shift σ is given by 6.1.

By $\mathbb{Z}_p$ we mean the group of integers $\{0, 1, \dots, p-1\}$ with addition mod p. The G action on X determines a homomorphism $\phi_X \colon G \to \mathbb{Z}_p$, given by $\phi_X(g) = k$ if and only if $g \colon (\overline{X}, 0) \mapsto (\overline{X}, k)$. We refer to ϕ_X as the *action homomorphism* for the G-SFT (X, σ). Note that for $0 \le n \le p-1$ and for each $g \in G$, $g \colon (\overline{X}, n) \mapsto (\overline{X}, n + \phi_X(g) (\mathrm{mod}\ p))$, where the action on the first coordinate is given by some automorphism U_g of $(\overline{X}, \overline{\sigma})$. The first coordinate automorphisms $\{U_g\}_{g \in G}$ define a G action on $(\overline{X}, \overline{\sigma})$, given by $g \colon \overline{x} \mapsto U_g(\overline{x})$. This G action on $\overline{X}$ is not necessarily free, even if the G action on X is free. We refer to the G-SFT $\overline{X}$ as the *base* G-SFT for X. We point out that base G-SFTs are mixing, so right closing almost conjugacy of base G-SFTs is classified by Theorem 5.2.

Theorem 6.2. *Let X and Y be irreducible G-SFTs. Then the following are equivalent.*

(1) *X and Y are right closing almost conjugate as G-SFTs.*

(2) *The base G-SFTs* $\overline{X}$ *and* $\overline{Y}$ *for* X *and* Y *are right closing almost conjugate as G-SFTs, and the action homomorphisms* ϕ_X *and* ϕ_Y *are the same.*

Proof. Suppose (X, σ) and (Y, σ) are right closing almost conjugate as G-SFTs. Then there is a G-SFT (Z, σ) and 1-1 *a.e.* right closing G-factor maps $\pi_X \colon Z \to X$ and $\pi_Y \colon Z \to Y$. The maps π_X and π_Y preserve period, so Z must have period p, where p is the period of both X and Y. Furthermore Z must be irreducible because X and Y are irreducible. Without loss of generality, assume that $Z = \overline{Z} \times \{0, \ldots, p-1\}$ where $\overline{Z}$ is the base G-SFT for Z. Further assume $(\overline{X}, 0) = \pi_X(\overline{Z}, 0)$ and $(\overline{Y}, 0) = \pi_Y(\overline{Z}, 0)$, where $\overline{X}$ and $\overline{Y}$ are the base G-SFTs for X and Y respectively. Observe that for $0 \le n \le p-1$,

$$\begin{aligned} \pi_X(\overline{Z}, n) &= \pi_X \circ \sigma^n(\overline{Z}, 0) \\ &= \sigma^n \circ \pi_X(\overline{Z}, 0) \\ &= \sigma^n(\overline{X}, 0) \\ &= (\overline{X}, n). \end{aligned}$$

In particular $\phi_X = \phi_Z$ (since π_X intertwines G actions). Similarly $\phi_Y = \phi_Z$.

Let $P_Z \colon Z \to \overline{Z}$ be the G-factor map $(\overline{z}, n) \mapsto \overline{z}$ and let $P_X \colon X \to \overline{X}$ be the G-factor map $(\overline{x}, n) \mapsto \overline{x}$. Since $\pi_X(\overline{Z}, n) = (\overline{X}, n)$ for $0 \le n \le p-1$, there is a G-factor map $\overline{\pi}_X \colon \overline{Z} \to \overline{X}$ which makes the following diagram commute.

$$\begin{array}{ccc} Z & \xrightarrow{\pi_X} & X \\ {\scriptstyle P_Z}\downarrow & & \downarrow{\scriptstyle P_X} \\ \overline{Z} & \xrightarrow{\overline{\pi}_X} & \overline{X} \end{array}$$

$\overline{\pi}_X$ is 1-1 *a.e.* and right closing because π_X is. Similarly construct a 1-1 *a.e.* right closing G-factor map $\overline{\pi}_Y \colon \overline{Z} \to \overline{Y}$. Then $\overline{X}$ and $\overline{Y}$ are right closing almost conjugate as G-SFTs.

Conversely, suppose the base G-SFTs $(\overline{X}, \overline{\sigma})$ and $(\overline{Y}, \overline{\sigma})$ are right closing almost conjugate as G-SFTs, and $\phi = \phi_X = \phi_Y$. In particular, X and Y have the same period p. Let $(\overline{Z}, \overline{\sigma})$ be a G-SFT with 1-1 *a.e.* right closing G-factor maps $\overline{\pi}_X \colon \overline{Z} \to \overline{X}$ and $\overline{\pi}_Y \colon \overline{Z} \to \overline{Y}$. Let $Z = \overline{Z} \times \{0, \ldots, p-1\}$ with the shift defined as in 6.1. Define a G action on Z by $g \colon (\overline{z}, n) \mapsto (\overline{z} \cdot g, n + \phi(g) (\mathrm{mod}\ p))$. Define maps $\pi_X \colon Z \to X$ and $\pi_Y \colon Z \to Y$ by $\pi_X(\overline{z}, n) = (\overline{\pi}_X(\overline{z}), n)$ and $\pi_Y(\overline{z}, n) = (\overline{\pi}_Y(\overline{z}), n)$. Then π_X and π_Y are G-factor maps. They are 1-1 *a.e.* and right closing because $\overline{\pi}_X$ and $\overline{\pi}_Y$ are. □

7. Regular isomorphism of G-Markov chains

Let (X, μ) and (Y, ν) be irreducible Markov chains with Markov measures μ and ν. Let α and β be the time zero partitions of X and Y, respectively.

Consider the past σ-algebras $\alpha^- = \bigvee_{n=0}^{\infty} \sigma^n \alpha$ and $\beta^- = \bigvee_{n=0}^{\infty} \sigma^n \beta$. Then (X, μ) and (Y, ν) are *regularly isomorphic* if there is a measurable isomorphism $\phi\colon (X, \mu) \to (Y, \nu)$ such that

$$\phi^{-1}(\beta^-) \subset \sigma^{-N} \alpha^- = \alpha^- \vee \sigma^{-1}\alpha \vee \cdots \vee \sigma^{-N}\alpha \,,$$
$$\phi(\alpha^-) \subset \sigma^{-N} \beta^- = \beta^- \vee \sigma^{-1}\beta \vee \cdots \vee \sigma^{-N}\beta$$

for some non-negative integer N. The idea of regular isomorphism was introduced and studied by Parry, first in [9] and also in [14]. For a regular isomorphism ϕ (in contrast to an arbitrary measurable isomorphism), to code the present $(\phi x)_0$, it suffices to know the past and a bounded look into the future $x_{(-\infty,N]}$. Boyle and Tuncel [8] show this measurable coding relation has a more finite and continuous formulation, as follows.

Theorem 7.1. *Irreducible Markov chains (X, μ) and (Y, ν) are regularly isomorphic if and only if there exists an irreducible Markov chain (Z, η) and* 1-1 *a.e. right closing factor maps $\pi_X \colon (Z, \eta) \to (X, \mu)$ and $\pi_Y \colon (Z, \eta) \to (Y, \nu)$.*

A *G-Markov chain* is a Markov chain (X, μ) such that X is a G-SFT and μ is a G-invariant Markov measure on X. Say that irreducible G-Markov chains (X, μ) and (Y, ν) are *G-regularly isomorphic* if there is a regular isomorphism $\phi\colon (X, \mu) \to (Y, \nu)$ such that ϕ is G-equivariant. By Theorems 4.1 and 7.1 we have the following.

Corollary 7.2. *Mixing free G-Markov chains (X, μ_X) and (Y, μ_Y), with unique measures of maximal entropy μ_X and μ_Y, are G-regularly isomorphic if and only if (X, μ_X) and (Y, μ_Y) are regularly isomorphic as Markov chains.*

In the general irreducible case, G-regular isomorphism with respect to measures of maximal entropy can be classified in terms of the invariants of Theorem 6.2.

References

[1] J. Ashley, *Bounded-to-1 factors of an aperiodic shift of finite type are 1-to-1 almost everywhere factors also*, Ergodic Theory & Dynamical Systems **10** (1990), 615-625.

[2] R. Adler, B. Kitchens and B. Marcus, *Finite group actions on shifts of finite type*, Ergodic Theory & Dynamical Systems **5** (1985), 1-25.

[3] R. Adler, B. Kitchens and B. Marcus, *Almost topological classification of finite-to-one factor maps between shifts of finite type*, Ergodic Theory & Dynamical Systems **5** (1985), 485-500.

[4] R. Adler and B. Marcus, *Topological entropy and equivalence of dynamical systems*, Mem. Amer. Math. Soc., **20**, No. 219 (1979).

[5] M. Boyle and D. Handelman, *The spectra of nonnegative matrices via symbolic dynamics*, Annals of Math. **133** (1991), 249-316.

[6] M. Boyle, B. Marcus and P. Trow, *Resolving maps and the dimension group for shifts of finite type*, Mem. Amer. Math. Soc. **70** (1987).

[7] M. Boyle and M. Sullivan, *Equivariant flow equivalence for shifts of finite type, by matrix equivalence over group rings*, Proc. London Math. Soc. **91** (2005), 184-214.

[8] M. Boyle and S. Tuncel, *Regular isomorphism of Markov chains is almost topological*, Ergodic Theory & Dynamical Systems **10** (1990), 89-100.

[9] R. Fellgett and W. Parry, *Endomorphisms of Lebesgue space II*, Bull. London Math. Soc. **7** (1975), 151-158.

[10] M. Field and M. Nicol, *Ergodic theory of equivariant diffeomorphisms: Markov partitions and stable ergodicity*, Mem. Amer. Math. Soc. **169**, No. 803 (2004).

[11] B. Kitchens, *Symbolic dynamics: one-sided, two-sided and countable state Markov shifts*, Springer (1991).

[12] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge Univ. Press (1995).

[13] M. Nasu, *Topological conjugacy for sofic systems and extensions of automorphisms of finite subsystems of topological Markov shifts*, Springer Lec. Notes in Math **1342** (1988), 564-607.

[14] W. Parry, *Notes on coding problems for finite state processes*, Bull. London Math. Soc. **23** No. 1 (1991), 1-33.

[15] W. Parry, *Unpublished communication*, 2003.

[16] W. Parry, *The Livšic periodic point theorem for non-abelian cocycles*, Ergodic Theory & Dynamical Systems **19** (1999), 687-701.

[17] K. Schmidt, *Remarks on Livšic' theory for nonabelian cocycles*, Ergodic Theory & Dynamical Systems **19** (1999), 703-721.

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