

# WEAK EQUIVALENCE FOR SHIFTS OF FINITE TYPE

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ABSTRACT. We derive a computable set of necessary and sufficient conditions for the existence of a homomorphism from one shift of finite type to another. Also we consider an equivalence relation on subshifts, called weak equivalence, which was introduced and studied by Beal and Perrin. We classify arbitrary shifts of finite type up to weak equivalence.

## 1. INTRODUCTION

In [2], Beal and Perrin introduce an equivalence relation on subshifts called weak equivalence, and classify a special collection of irreducible shifts of finite type (SFTs) up to weak equivalence. We classify arbitrary SFTs up to weak equivalence. For this, first we introduce the notion of a *phase matrix* for an SFT, and show that the set of phase matrices for an SFT is a computable isomorphism invariant. With Theorem 4.5 we formulate necessary and sufficient conditions in terms of phase matrices for there to exist an extension of a given homomorphism  $NW(S) \rightarrow NW(T)$  between nonwandering sets of SFTs  $S$  and  $T$  to a homomorphism  $S \rightarrow T$ . We use this extension result to give necessary and sufficient conditions for the existence of a homomorphism from one SFT to another (Theorem 5.1).

Extension results are significant in symbolic dynamics. Boyle's extension result [3, Extension Lemma 2.4] is the key to characterizing when there exists an epimorphism from one mixing SFT onto another of lower entropy, and has had other applications as well. (For example, a refinement of that result, [5, Theorem 5.3], establishes that the Markovian property of a homomorphism is in a certain sense not local.) The extension theorem for inert automorphisms of Kim and Roush [6] is a central tool in the study of automorphisms of an SFT. (See [4], [7] and [8].) The study [1] provides a variety of surjective extension results. It follows from the work of Maass [13] that an open problem on extensions, Problem 6.8 below, is very closely related to the problem of characterizing the limit sets of stable cellular automaton maps.

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As a consequence of Lightwood's machinery for extending certain classes of homomorphisms of  $\mathbb{Z}^2$  SFTs, we know that for a large class of examples the key issue for existence of an embedding between  $\mathbb{Z}^2$  SFTs is the existence of a homomorphism between them [10, 11]. This provides some additional motivation for our existence result in the more tractable  $d = 1$  case.

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## 2. PRELIMINARIES

**2.1. Shifts of Finite Type.** We assume some familiarity with shifts of finite type. See [9] and [12] for more complete background.

Given  $n > 0$ , let  $\Sigma_n = \{0, \dots, n-1\}^{\mathbb{Z}}$  denote the set of doubly-infinite sequences  $(x_i)_{i \in \mathbb{Z}}$  where each  $x_i \in \{0, \dots, n-1\}$ . Give to  $\Sigma_n$  the product of the discrete topology on  $\{0, \dots, n-1\}$ . Then  $\Sigma_n$  is compact and metrizable. The *shift* is the homeomorphism  $\sigma : \Sigma_n \rightarrow \Sigma_n$  given by  $\sigma(x)_i = x_{i+1}$ . The pair  $(\Sigma_n, \sigma)$  (or just  $\Sigma_n$  for short) is the *full  $n$ -shift*. A *full shift* is a full  $n$ -shift for some  $n$ . If  $S$  is a closed,  $\sigma$ -invariant subset of a full shift, then the pair  $(S, \sigma)$  (or just  $S$  for short) is a *subshift*.

A *word* on  $\{0, \dots, n-1\}$  is a finite concatenation  $w_1 \dots w_k$ , where each  $w_i \in \{0, \dots, n-1\}$ . Given a subshift  $S$ , a point  $x \in S$ , and coordinates  $i < j$ , often we write  $x[i, j]$  to denote the word  $x_i \dots x_j$ .

If a subshift  $S$  may be written

$$S = \Sigma_n \setminus \{x \in \Sigma_n : \text{no word in } \mathcal{F} \text{ appears in } x\}$$

for a finite set of words  $\mathcal{F}$ , then  $S$  is a *shift of finite type (SFT)*.

A *homomorphism*  $f : S \rightarrow T$  from one SFT to another is a  $\sigma$ -equivariant function such that, for each  $x \in S$ ,  $f(x)_0$  is determined by  $x_{-i} \dots x_0 \dots x_j$  for some  $i, j \geq 0$ . Then  $f$  is  *$r$ -block* where  $r = i + j + 1$ . If  $f$  is surjective it is an *epimorphism*. An injective epimorphism is an *isomorphism*. Any homomorphism  $f : S \rightarrow T$  is isomorphic to a one-block homomorphism, in the sense that there exists an SFT  $S'$  and an isomorphism  $\varphi : S' \rightarrow S$  such that  $f \circ \varphi$  is one-block.

**2.2. Edge Shifts.** Let  $A$  be an  $n \times n$  matrix over the non-negative integers which has no 0-row or 0-column. Then  $A$  is the adjacency matrix for a directed graph  $\mathcal{G}(A)$ :  $\mathcal{G}(A)$  has vertex set  $\{v_1, \dots, v_n\}$  where the number of edges from  $v_i$  to  $v_j$  is  $A(i, j)$ . Let  $\mathcal{E}(A) = \{\text{edges in } \mathcal{G}(A)\}$  and  $\mathcal{V}(A) = \{\text{vertices in } \mathcal{G}(A)\}$ . Put

$$\Sigma_A = \{x = (x_i)_{i \in \mathbb{Z}} \in \mathcal{E}(A)^{\mathbb{Z}} \mid \text{each } x_i x_{i+1} \text{ is a path in } \mathcal{G}(A)\}.$$

Give to  $\Sigma_A$  the relative of the product of the discrete topology on  $\mathcal{E}(A)$ . Then, by thinking of edges in  $\mathcal{E}(A)$  as numbers in  $\{0, \dots, |\mathcal{E}(A)| - 1\}$ ,  $\Sigma_A$  is a special type of subshift, called an *edge shift*. Any edge shift is an SFT. Also, any SFT is isomorphic to an edge shift.

**2.3. Irreducible Shifts of Finite Type.** Let  $A$  be as in section 2.2. Then  $A$ ,  $\mathcal{G}(A)$ , and  $\Sigma_A$  are *irreducible* if, for each  $1 \leq i, j \leq n$ , there exists  $L = L(i, j)$  such that  $A^L(i, j) > 0$ . If there exists a uniform  $L > 0$  such that  $A^L(i, j) > 0$  for all  $1 \leq i, j \leq n$ , then  $A$  and  $\mathcal{G}(A)$  are *primitive*. If  $A$  is primitive, then  $\Sigma_A$  is *mixing*.

A point  $x \in \Sigma_A$  is *periodic* if there exists  $p > 0$  such that  $\sigma^p(x) = x$ . In this case  $p$  is a *period* of  $x$ . When  $A$  is irreducible, we define the period of  $A$ ,  $\mathcal{G}(A)$  and  $\Sigma_A$  to be the greatest common divisor of the set of periods of the periodic points in  $\Sigma_A$ . Note that for primitive  $A$ , the period of  $A$  is 1.

Given a path  $w = w_0 \cdots w_{l-1}$  of edges in  $\mathcal{G}(A)$ , the *length* of  $w$  is  $|w| = l$ . Denote by  $\text{in}(w)$  and  $\text{ter}(w)$  the initial and terminal vertices of  $w$ , respectively. If  $A$  is irreducible, say that vertices  $v_i$  and  $v_j$  in  $\mathcal{V}(A)$  are *period equivalent* if there is a path  $w$  in  $\mathcal{G}(A)$  with  $\text{in}(w) = v_i$  and  $\text{ter}(w) = v_j$  such that  $|w|$  is divisible by  $p =$  the period of  $A$ . This defines an equivalence relation on  $\mathcal{V}(A)$ , and induces a partition of  $\mathcal{V}(A)$  into a disjoint union of  $p$  sets

$$\mathcal{V}(A) = \mathcal{V}^0(A) \sqcup \cdots \sqcup \mathcal{V}^{p-1}(A).$$

The  $\mathcal{V}^i(A)$  are called the *cyclically moving vertex sets* of  $\mathcal{V}(A)$ . This partition of  $\mathcal{V}(A)$  induces a partition of  $\Sigma_A$  into  $p$  sets

$$\Sigma_A = \Sigma_A^0 \sqcup \cdots \sqcup \Sigma_A^{p-1},$$

where  $\Sigma_A^i = \{x \in \Sigma_A : \text{in}(x_0) \in \mathcal{V}^i(A)\}$ . The  $\Sigma_A^i$  are called the *cyclically moving subsets* of  $\Sigma_A$ . By re-labelling if necessary, we may assume that  $\sigma(\Sigma_A^i) = \Sigma_A^{(i+1) \bmod p}$  for  $0 \leq i \leq p-1$ . Each  $\Sigma_A^i$  is both invariant and mixing under  $\sigma^p$ .

**2.4. Reducible Shifts of Finite Type.** Let  $A$  be as in section 2.2. The *nonwandering* set of  $\Sigma_A$ , denoted  $NW(A)$ , is the closure of the set of periodic points in  $\Sigma_A$ . The nonwandering set  $NW(A)$  is an SFT, and is a disjoint union

$$NW(A) = C(A)_1 \sqcup \cdots \sqcup C(A)_K,$$

where each  $C(A)_i$  is an irreducible SFT. The  $C(A)_i$  are called the *irreducible components* of  $\Sigma_A$ . If  $K > 1$ , then  $\Sigma_A$  is *reducible*.

For  $1 \leq i \leq K$ , let  $p_i$  denote the period of  $C(A)_i$ . Then  $C(A)_i$  is partitioned into cyclically moving subsets  $C(A)_i^0, \dots, C(A)_i^{p_i-1}$ , each

of which is both invariant and mixing under  $\sigma^{p_i}$ , as in Section 2.3. Therefore we can find  $l_i \geq 1$  such that if  $q$  and  $q'$  are any two vertices in the same cyclically moving subset of the graph  $\mathcal{G}(C(A)_i)$ , then there exists a path  $w$  in  $\mathcal{G}(C(A)_i)$  of length  $|w| = p_i \cdot l_i$  such that  $\text{in}(w) = q$  and  $\text{ter}(w) = q'$ . Such  $l_i$  is called a *cyclic transition length* for  $C(A)_i$ .

### 3. HOMOMORPHISMS BETWEEN IRREDUCIBLE SFTS

Let  $S$  and  $T$  be subshifts. Write  $S \xrightarrow{PER} T$  if, for each periodic point  $x \in S$  of period  $p$ , there is a periodic point  $y \in T$  of period  $q$ , such that  $q$  divides  $p$ .

**Theorem 3.1.** *Given irreducible SFTs  $S$  and  $T$ , there exists a homomorphism  $f : S \rightarrow T$  if and only if  $S \xrightarrow{PER} T$ . Moreover, assuming  $S \xrightarrow{PER} T$ , then given any pair of cyclically moving subsets  $S^0$  and  $T^0$  in  $S$  and  $T$  respectively,  $f : S \rightarrow T$  may be chosen so that  $f(S^0) \subset T^0$ .*

*Proof.* If  $f : S \rightarrow T$  is a block code and  $x \in S$  is a periodic point of period  $p$ , then  $f(x) \in T$  must be a periodic point of some period dividing  $p$ , by the  $\sigma$ -equivariance of  $f$ .

Conversely, first suppose  $S \xrightarrow{PER} T$  and  $T$  is mixing. In this case, let  $\bar{S}$  be the empty subshift, and let  $\bar{f} : \bar{S} \rightarrow T$  be the empty homomorphism. By [3, Extension Lemma 2.4],  $\bar{f}$  extends to a homomorphism  $f : S \rightarrow T$ .

Now suppose  $S \xrightarrow{PER} T$  where  $T$  is no longer assumed to be mixing. Decompose  $S$  and  $T$  into disjoint unions of cyclically moving subsets, each of which is invariant and mixing under the  $p^{\text{th}}$  power of the shift, where  $p$  is the period of  $S$  (the period of  $T$  divides  $p$  by assumption). Fix cyclically moving subsets  $S^0$  of  $S$  and  $T^0$  of  $T$ , and let  $X = (S^0, \sigma^p)$  and  $Y = (T^0, \sigma^p)$ . Then  $X \xrightarrow{PER} Y$  and  $Y$  is mixing so, by the previous paragraph, there exists a homomorphism  $f_0 : X \rightarrow Y$ . Every other cyclically moving subset of  $S$  is equal to  $\sigma^i(S^0)$  for some unique  $1 \leq i < p$ . So define a homomorphism  $f_i$  on  $\sigma^i(S^0)$  by  $f_i = \sigma^i \circ f_0 \circ \sigma^{-i}$ , and define  $f : S \rightarrow T$  by  $f(x) = f_i(x)$  where  $x \in \sigma^i(S^0)$ . □

### 4. EXTENDING HOMOMORPHISMS DEFINED ON A NONWANDERING SET

Let  $\Sigma_A$  be a reducible edge shift. Write the nonwandering set  $NW(A) = C(A)_1 \sqcup \cdots \sqcup C(A)_K$  as a disjoint union of irreducible components, and let  $p_i$  denote the period of  $C(A)_i$ . Fix in each  $C(A)_i$  one cyclically

moving subset  $C(A)_i^0$ , and enumerate the remaining cyclically moving subsets in  $C(A)_i$  by  $C(A)_i^1, \dots, C(A)_i^{p_i-1}$ , where  $C(A)_i^k = \sigma^k(C(A)_i^0)$ .

Given this enumeration of the cyclically moving subsets, for each  $i, j$ , define the set of *connection paths* from  $C(A)_i$  to  $C(A)_j$ , denoted  $CP_A(i, j)$ , to be the set of paths in  $\mathcal{G}(A)$  of the form  $x_0 \cdots x_{t-1}$  which have initial vertex in some  $C(A)_i^s$ , and terminal vertex in some  $C(A)_j^r$ . The number  $s + t - r$ , taken mod  $\gcd(p_i, p_j)$ , is the *phase change* of the path  $x_0 \cdots x_{t-1}$ .

Now define a  $K \times K$  matrix  $P_A$  as follows. The entries of  $P_A$  are sets. Specifically,  $P_A(i, j) \subset \mathbb{Z}_{\gcd(p_i, p_j)}$  is the set of phase changes of paths in  $CP_A(i, j)$ . Let  $\mathcal{M}(A)$  denote the set of phase matrices for  $A$ . Note that the finitely many possible enumerations of the cyclically moving subsets determine the finitely many matrices in  $\mathcal{M}(A)$ . By a *phase matrix* for  $A$  we mean a matrix  $P_A$  determined by some enumeration of cyclically moving subsets.

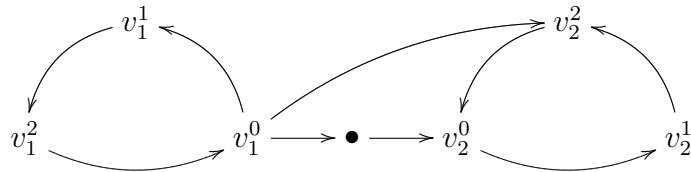
Given  $x$  and  $x'$  in  $\Sigma_A$ , say that  $x$  and  $x'$  are *backwardly asymptotic* if there exists  $k \in \mathbb{Z}$  such that  $x_i = x'_i$  for all  $i \leq k$ , and *forwardly asymptotic* if there exists  $k \in \mathbb{Z}$  such that  $x_i = x'_i$  for all  $i \geq k$ .

**Remark 4.1.** Let  $P_A$  be a phase matrix for  $A$ . Then  $c \in P_A(i, j)$  if and only if there exist  $x, y$  and  $z$  in  $\Sigma_A$  such that

- $z$  is backwardly asymptotic to  $x$  and forwardly asymptotic to  $y$ ,
- $x \in C(A)_i^s$  and  $y \in C(A)_j^r$ , and
- $s - r \equiv c \pmod{\gcd(p_i, p_j)}$ .

It follows that, if  $\phi : \Sigma_A \rightarrow \Sigma_C$  is an isomorphism which respects the numberings of the irreducible components and cyclically moving subsets, then the associated phase matrices  $P_A$  and  $P_C$  are equal. In particular, the set  $\mathcal{M}(A)$  of possible phase matrices for  $A$  is an isomorphism invariant.

**Example 4.2.** Let  $A$  be the adjacency matrix for the following graph.



Let  $C(A)_1$  correspond to the cycle on the left and let  $C(A)_2$  correspond to the cycle on the right. Choose the cyclically moving subset  $C(A)_1^0$  to be those points  $x \in C(A)_1$  such that the initial vertex of  $x_0$

is  $v_1^0$ , and  $C(A)_2^0$  to be those points  $x \in C(A)_2$  such that the initial vertex of  $x_0$  is  $v_2^0$ . Then the phase matrix for this choice is

$$P_A = \begin{pmatrix} \{0\} & \{2\} \\ \emptyset & \{0\} \end{pmatrix}.$$

**Remark 4.3.** Let  $\mathcal{D}(A)$  denote the set of  $K \times K$  diagonal matrices  $D$ , where each  $D(i, i)$  is an element of the group  $\mathbb{Z}_{p_i}$ . Note that, if  $M$  and  $M'$  are any two phase matrices in  $\mathcal{M}(A)$ , then there is a matrix  $D \in \mathcal{D}(A)$  such that  $M' = DMD^{-1}$ , by which we mean that, for each  $i, j$ ,

$$(4.4) \quad M'(i, j) = \{D(i, i) + s - D(j, j) : s \in M(i, j)\}.$$

Conversely, given  $M \in \mathcal{M}(A)$  and  $D \in \mathcal{D}(A)$ , the matrix  $DMD^{-1}$  is in  $\mathcal{M}(A)$ . Thus the set  $\mathcal{M}(A)$  is computable via the following finite process:

- (1) Arbitrarily choose an enumeration of the cyclically moving subsets for  $A$ .
- (2) Construct the phase matrix  $P_A$  for this choice, as defined above, and include this  $P_A$  in  $\mathcal{M}(A)$ .
- (3) For each  $D \in \mathcal{D}(A)$ , add to  $\mathcal{M}(A)$  the matrix  $DP_AD^{-1}$ .

Now let  $\Sigma_B$  be another reducible edge shift with nonwandering set  $NW(B) = C(B)_1 \sqcup \cdots \sqcup C(B)_L$ , and let  $q_i$  denote the period of the irreducible component  $C(B)_i$ . Suppose there exists a homomorphism  $f_0 : NW(A) \rightarrow NW(B)$ . Then  $f_0$  induces a set function  $g : \{1, \dots, K\} \rightarrow \{1, \dots, L\}$  by the following rule:  $C(B)_{g(i)}$  is the irreducible component of  $\Sigma_B$  which contains  $f_0(C(A)_i)$ .

In each irreducible component  $C(B)_i$ , arbitrarily choose one cyclically moving subset  $C(B)_i^0$ , and enumerate the remaining cyclically moving subsets in  $C(B)_i$  by  $C(B)_i^k = \sigma^k(C(B)_i^0)$ . Let  $P_B$  be the phase matrix determined by this choice. Say that a phase matrix  $P_A$  for  $A$  is *compatible* with  $(P_B, f_0)$  if  $P_A$  is the phase matrix determined by a choice of the cyclically moving subsets  $C(A)_i^0$  in  $C(A)_i$  such that each  $f_0(C(A)_i^0) \subset C(B)_{g(i)}^0$ . For such  $P_A$  let  $\bar{P}_A$  be the matrix such that each  $\bar{P}_A(i, j)$  is the set of elements in  $P_A(i, j)$  taken mod  $\gcd(q_{g(i)}, q_{g(j)})$ .

**Theorem 4.5.** *Let  $\Sigma_A$  and  $\Sigma_B$  be reducible edge shifts, as above, and let  $f_0 : NW(A) \rightarrow NW(B)$  be a homomorphism. Let  $g : \{1, \dots, K\} \rightarrow \{1, \dots, L\}$  be the set function induced by  $f_0$ . Choose a phase matrix  $P_B$  for  $B$ , and let  $P_A$  be a phase matrix for  $A$  compatible with  $(P_B, f_0)$ .*

Then  $f_0$  extends to a homomorphism  $f : \Sigma_A \rightarrow \Sigma_B$  if and only if each  $\overline{P}_A(i, j)$  is contained in  $P_B(g(i), g(j))$ .

*Proof.* Suppose  $f_0$  extends to  $f : \Sigma_A \rightarrow \Sigma_B$ . Let  $\bar{c} \in \overline{P}_A(i, j)$  and let  $c \in P_A(i, j)$  satisfy  $c \equiv \bar{c} \pmod{\gcd(q_{g(i)}, q_{g(j)})}$ . By Remark 4.1 there exist  $x, y$  and  $z$  in  $\Sigma_A$  such that

- $z$  is backwardly asymptotic to  $x$  and forwardly asymptotic to  $y$ ,
- $x \in C(A)_i^s$  and  $y \in C(A)_j^r$ , and
- $s - r \equiv c \pmod{\gcd(p_i, p_j)}$ .

Let  $\bar{s}$  ( $\bar{r}$ , respectively) denote  $s$  ( $r$ , resp.) taken modulo  $q_{g(i)}$  ( $q_{g(j)}$ , resp.). Then

- $f(z)$  is backwardly asymptotic to  $f(x)$  and forwardly asymptotic to  $f(y)$ ,
- $f(x) \in C(B)_{g(i)}^{\bar{s}}$  and  $f(y) \in C(B)_{g(j)}^{\bar{r}}$ , and
- $\bar{s} - \bar{r} \equiv \bar{c} \pmod{\gcd(q_{g(i)}, q_{g(j)})}$ .

Hence  $\bar{c} \in P_B(g(i), g(j))$ .

For the converse first note that we may assume, without loss of generality, that  $f_0$  is one-block. For, if not, then we can choose an SFT  $\Sigma_C$  and an isomorphism  $\varphi : \Sigma_C \rightarrow \Sigma_A$  such that  $f_0 \circ \varphi|_{NW(C)}$  is one-block. The isomorphism  $\varphi$  can be taken to respect the numberings of the irreducible components and cyclically moving subsets, which implies  $P_C = P_A$ , by Remark 4.1. It follows that  $f_0$  extends to all of  $\Sigma_A$  if and only if  $f_0 \circ \varphi|_{NW(C)}$  extends to all of  $\Sigma_C$ .

**Claim 4.6.** *Suppose each  $\overline{P}_A(i, j) \subset P_B(g(i), g(j))$ . Then there exists  $M > 0$  such that, for each  $i, j$  and for each path  $Q \in CP_A(i, j)$  of length  $|Q| \geq M$ , there exists a path  $Q' \in CP_B(g(i), g(j))$  such that  $|Q'| = |Q|$ ,  $\text{in}(Q') = f_0(\text{in}(Q))$ , and  $\text{ter}(Q') = f_0(\text{ter}(Q))$ .*

To prove Claim 4.6, first recall that each  $C(A)_i$  has a cyclic transition length  $l_i$ . WLOG assume that  $l_i$  is a cyclic transition length for  $C(B)_{g(i)}$  as well. (If not, just make  $l_i$  larger.) Let

$$l = \max_i \{l_i\},$$

and let

$$p = \prod_i p_i.$$

Note that if  $v$  and  $v'$  are any two vertices in the same cyclically moving subset of some  $C(B)_{g(i)}$ , then there is a path in  $C(B)_{g(i)}$  of length  $lp$  from  $v$  to  $v'$ .

Next, pick  $T$  large enough so that, for each  $i, j$  and for each  $\bar{c} \in \bar{P}_A(i, j)$ , there exists a path  $Q \in CP_B(g(i), g(j))$  of length at most  $T$  with phase change  $\bar{c}$ .

Then, for each  $i, j$ , pick  $R_{i,j}$  large enough so that all integers  $r$  greater than  $R_{i,j}$  and divisible by  $\gcd(q_{g(i)}, q_{g(j)})$  are contained in the  $\mathbb{N}$ -ideal

$$\langle q_{g(i)}, q_{g(j)} \rangle_{\mathbb{N}} := \{mq_{g(i)} + nq_{g(j)} : m \in \mathbb{N}, n \in \mathbb{N}\}.$$

Let  $R = \max_{i,j} R_{i,j}$ , and set

$$M = T + R + 2lp + 2 \max_i q_{g(i)}.$$

To see that  $M$  satisfies the requirements of Claim 4.6, let  $Q \in CP_A(i, j)$  have length  $|Q| \geq M$  and phase change  $c$ . Assume that  $\text{in}(Q)$  is in some  $C(A)_i^s$  and  $\text{ter}(Q)$  is in some  $C(A)_j^r$ , so that  $c \equiv s + |Q| - r \pmod{\gcd(p_i, p_j)}$ . Let  $Q_1 \in CP_B(g(i), g(j))$  have phase change  $\bar{c} \equiv c \pmod{\gcd(q_{g(i)}, q_{g(j)})}$  and  $|Q_1| \leq T$ .

Extend  $Q_1$  to the left in  $\mathcal{G}(C(B)_{g(i)})$  by at most  $q_{g(i)}$  and to the right in  $\mathcal{G}(C(B)_{g(j)})$  by at most  $q_{g(j)}$  to a path  $Q_2 \in CP_B(g(i), g(j))$  with  $\text{in}(Q_2) \in C(B)_{g(i)}^{\bar{s}}$  and  $\text{ter}(Q_2) \in C(B)_{g(j)}^{\bar{r}}$ , where  $\bar{s} \equiv s \pmod{q_{g(i)}}$  and  $\bar{r} \equiv r \pmod{q_{g(j)}}$ . Note that  $Q_2$  has phase change  $\bar{c}$ . Also,  $|Q| - 2lp - |Q_2|$  is divisible by  $\gcd(q_{g(i)}, q_{g(j)})$  and greater than  $R$ , so

$$|Q| - 2lp - |Q_2| \in \langle q_{g(i)}, q_{g(j)} \rangle_{\mathbb{N}}.$$

Hence

$$|Q| - 2lp = |Q_2| + aq_{g(i)} + bq_{g(j)}$$

for some  $a, b \geq 0$ . So we may extend  $Q_2$  to the left in  $\mathcal{G}(C(B)_{g(i)})$  by  $aq_{g(i)}$  and to the right in  $\mathcal{G}(C(B)_{g(j)})$  by  $bq_{g(j)}$  to a path  $Q_3 \in CP_B(g(i), g(j))$  with

- $\text{in}(Q_3) \in C(B)_{g(i)}^{\bar{s}}$ ,
- $\text{ter}(Q_3) \in C(B)_{g(j)}^{\bar{r}}$ , and
- $|Q| = |Q_3| + 2lp$ .

Finally, by choice of  $l$ , we may extend  $Q_3$  to the left in  $\mathcal{G}(C(B)_{g(i)})$  by  $lp$  and to the right in  $\mathcal{G}(C(B)_{g(j)})$  by  $lp$  to a path  $Q' \in CP_B(g(i), g(j))$  with

- $\text{in}(Q') = f_0(\text{in}(Q))$ ,
- $\text{ter}(Q') = f_0(\text{ter}(Q))$ , and
- $|Q| = |Q'|$ .

This completes the proof of Claim 4.6

Now define  $f : \Sigma_A \rightarrow \Sigma_B$  as follows. Let  $M > 0$  be as in Claim 4.6, and let  $Q \rightarrow Q'$  be a chosen associated map on connection paths  $Q$

of length at least  $M$ . Consider the set of paths  $VWX$  in  $\mathcal{G}(A)$  which satisfy

- (1)  $V$  is in a component  $C(A)_i$  and  $|V| = M$ ,
- (2)  $X$  is in a component  $C(A)_j$  and  $|X| = M$ ,
- (3) the only  $C(A)_i$ -vertex of  $W$  is its initial vertex, and the only  $C(A)_j$ -vertex of  $W$  is its terminal vertex, and
- (4)  $W$  does not contain any path of length  $M$  from  $NW(A)$ .

Let  $x \in \Sigma_A$ . For each  $k \in \mathbb{Z}$  such that  $x[k, k + |VWX| - 1]$  is a path satisfying (1)–(4) above, set  $f(x)[k, k + |VW| - 1] = (VW)'$ . Elsewhere in  $x$ , define  $f(x)_i = f_0(x)_i$ . As there is an upper bound on the lengths of paths  $VWX$  which satisfy (1)–(4) above,  $f$  is a homomorphism. By construction,  $f_0$  is the restriction of  $f$  to  $NW(A)$ . □

## 5. HOMOMORPHISMS BETWEEN REDUCIBLE SFTS

Our main theorem below characterizes when there exists a homomorphism between arbitrary SFTs.

**Theorem 5.1.** *Let  $\Sigma_A$  and  $\Sigma_B$  be reducible edge shifts with nonwandering sets  $NW(A) = C(A)_1 \sqcup \cdots \sqcup C(A)_K$  and  $NW(B) = C(B)_1 \sqcup \cdots \sqcup C(B)_L$ . Let  $P_B$  be a phase matrix for  $B$ . Then there exists a homomorphism  $f : \Sigma_A \rightarrow \Sigma_B$  if and only if we may choose a set function  $g : \{1, \dots, K\} \rightarrow \{1, \dots, L\}$  and a phase matrix  $P_A$  for  $A$  such that, for  $1 \leq i, j \leq K$ ,*

- (1)  $C(A)_i \xrightarrow{\text{PER}} C(B)_{g(i)}$ , and
- (2)  $\overline{P}_A(i, j) \subset P_B(g(i), g(j))$ .

*Proof.* Given a homomorphism  $f : \Sigma_A \rightarrow \Sigma_B$ , let  $C(B)_{g(i)}$  be the irreducible component of  $\Sigma_B$  containing  $f(C(A)_i)$ . This defines a set function  $g : \{1, \dots, K\} \rightarrow \{1, \dots, L\}$ . Then, for  $1 \leq i \leq K$ ,  $f|_{C(A)_i} : C(A)_i \rightarrow C(B)_{g(i)}$  is a homomorphism, so  $C(A)_i \xrightarrow{\text{PER}} C(B)_{g(i)}$ , by Theorem 3.1.

For  $1 \leq j \leq L$  and  $i \in g^{-1}(j)$ , choose  $C(A)_i^0$  to be a cyclically moving subset of  $C(A)_i$  such that  $f(C(A)_i^0) \subset C(B)_j^0$ . Let  $P_A$  be the phase matrix for  $A$  induced by this choice of the cyclically moving subsets  $C(A)_i^0$  in  $C(A)_i$ . Then  $P_A$  is compatible with  $(P_B, f_0)$ , where  $f_0 = f|_{NW(A)}$ . By Theorem 4.5 each  $\overline{P}_A(i, j) \subset P_B(g(i), g(j))$ .

Conversely suppose we may choose a set function  $g : \{1, \dots, K\} \rightarrow \{1, \dots, L\}$  and a phase matrix  $P_A$  for  $A$  such that conditions (1) and (2) of Theorem 5.1 are satisfied for  $1 \leq i, j \leq K$ . By condition (1) and Theorem 3.1, there exists a homomorphism  $f_0 : NW(A) \rightarrow NW(B)$  such

that  $C(B)_{g(i)}$  is the irreducible component of  $\Sigma_B$  containing  $f_0(C(A)_i)$ . Moreover, by Theorem 3.1,  $f_0$  may be chosen so that each  $C(B)_{g(i)}^0$  contains  $f_0(C(A)_i^0)$  (i.e.  $P_A$  is compatible with  $(P_B, f_0)$ ). It then follows from Theorem 4.5 that  $f_0$  extends to a homomorphism  $f : \Sigma_A \rightarrow \Sigma_B$ .  $\square$

## 6. WEAK EQUIVALENCE OF SFTs

**Definition 6.1.** Let  $S$  and  $T$  be subshifts. If  $\varphi : \Sigma_n \rightarrow \Sigma_m$  is a homomorphism between full shifts  $\Sigma_n$  and  $\Sigma_m$  which contain  $S$  and  $T$  respectively, such that  $\varphi^{-1}(T) = S$ , then write  $\varphi : S \xrightarrow{\sim} T$ . If such a  $\varphi$  exists, write  $S \xrightarrow{\sim} T$ .

**Definition 6.2 (Beal and Perrin).** Subshifts  $S$  and  $T$  are *weak equivalent* if  $S \xrightarrow{\sim} T$  and  $T \xrightarrow{\sim} S$ .

A *flower shift* is an SFT presented by a directed graph made up of loops that all begin and end at a single vertex. If the lengths of the loops of a flower shift  $S$  are  $s := (s_1, \dots, s_n)$ , then we define  $\langle s \rangle_{\mathbb{N}}$  to be the  $\mathbb{N}$ -ideal  $s_1\mathbb{N} + \dots + s_n\mathbb{N}$ . Given flower shifts  $S$  and  $T$ , which define ideals  $\langle s \rangle_{\mathbb{N}}$  and  $\langle t \rangle_{\mathbb{N}}$ , Beal and Perrin [2] prove that  $S$  and  $T$  are weak equivalent if and only if  $\langle s \rangle_{\mathbb{N}} = \langle t \rangle_{\mathbb{N}}$ .

**Proposition 6.3.** *The following are equivalent for subshifts  $S$  and  $T$ .*

- (1)  $S \xrightarrow{\sim} T$ .
- (2) *There exists an SFT  $S'$  containing  $S$ , a subshift  $T'$  containing  $T$ , and a homomorphism  $f : S' \rightarrow T'$  such that  $f^{-1}(T) = S$ .*

*Proof.* (1)  $\Rightarrow$  (2): Let  $(f, S', T')$  be  $(\varphi, \Sigma_n, \Sigma_m)$  from Definition 6.1.

(2)  $\Rightarrow$  (1): Choose  $r \geq 2$  so that  $f$  is  $r$ -block. We may assume WLOG that  $S'$  is a 1-step SFT, which means that a point  $x$  is in  $S'$  if and only if  $x_i x_{i+1}$  is an allowed word of length 2 in  $S'$  for each  $i \in \mathbb{Z}$ . Let  $\Sigma_{n-1}$  and  $\Sigma_{m-1}$  be full shifts containing  $S'$  and  $T'$  respectively. Define an  $r$ -block homomorphism  $\varphi : \Sigma_n \rightarrow \Sigma_m$  by:

$$\varphi(x_0 \cdots x_{r-1}) = \begin{cases} f(x_0 \cdots x_{r-1}) & \text{if } x_0 \cdots x_{r-1} \text{ is allowed in } S' \\ m-1 & \text{if } x_0 \cdots x_{r-1} \text{ is not allowed in } S' \end{cases}$$

Then  $\varphi : \Sigma_n \rightarrow \Sigma_m$  is well-defined, and  $\varphi^{-1}(T) = S$ .  $\square$

**Theorem 6.4.** *Let  $S$  be an SFT and let  $T$  be a subshift. Then  $S \xrightarrow{\sim} T$  if and only if there exists a homomorphism  $f : S \rightarrow T$ .*

*Proof.* If  $\varphi : S \xrightarrow{\sim} T$ , then  $f = \varphi|_S$  is a homomorphism. Conversely, if  $f : S \rightarrow T$  is a homomorphism, then letting  $S' = S$  and  $T' = T$ , condition (2) of Proposition 6.3 is satisfied. Hence  $S \xrightarrow{\sim} T$ .  $\square$

The requirement that  $S$  be an SFT can not be removed from Theorem 6.4. To see this note that, if  $S$  is a non-SFT subshift and  $T$  is an SFT, then it can not be the case that  $S \xrightarrow{\sim} T$ . However there are plenty of examples of homomorphisms from non-SFT subshifts to SFT subshifts.

**Corollary 6.5.** *Let  $S$  and  $T$  be SFTs. Then  $S$  and  $T$  are weak equivalent if and only if there exist homomorphisms  $f : S \rightarrow T$  and  $g : T \rightarrow S$ .*

Corollary 6.5 follows directly from Theorem 6.4. Corollary 6.5 combined with Theorem 5.1 provide a classification of weak equivalence for SFTs.

The next proposition is an easy observation which we record to further emphasize the local nature of the weak equivalence relation. For a subshift  $S$ , let  $W_k(S)$  denote the set of words of length  $k$  occurring in points in  $S$ . Given  $n \in \mathbb{N}$ , let  $S_n$  denote the largest SFT such that  $W_n(S_n) = W_n(S)$ . If  $f : S \rightarrow T$  is an  $R$ -block homomorphism between subshifts then, for  $r \geq R$ , the  $R$ -block rule defining  $f$  also defines a homomorphism  $f_r : S_r \rightarrow T_{r-R}$ .

**Proposition 6.6.** *Suppose  $\varphi : S \xrightarrow{\sim} T$ , and let  $f = \varphi|_S$ . Then there exists a positive integer  $R$  such that, for all  $r \geq R$ ,  $f_r$  is well-defined and  $f_r^{-1}(T) = S$ .*

*Proof.* We have  $\varphi : \Sigma_n \rightarrow \Sigma_m$  for some  $n$  and  $m$ . Choose  $R$  so that  $\varphi$  is  $R$ -block. Observe that  $S \subset S_R \subset \Sigma_n$  because  $S_R \subset S_1$  and  $S_1$  is the smallest full shift containing  $S$ . Also  $\varphi|_{S_R} = f_R$  because  $\varphi$  and  $f$  are defined by the same  $R$ -block rule. Therefore  $\varphi|_{S_r} = f_r$  for all  $r \geq R$ . We know  $\varphi^{-1}(T) = S$ , so  $(\varphi|_{S_r})^{-1}(T) = S$  and hence  $f_r^{-1}(T) = S$  for all  $r \geq R$ .  $\square$

Say that a homomorphism  $f$  defined on a subshift  $S$  is *steady* if there is an SFT  $S'$  containing  $S$  such that  $f$  is well-defined on  $S'$ , and  $f$  applied to  $S'$  has the same image as does  $f$ . A subshift  $S$  is *sofic* if there exists an SFT  $R$  and an epimorphism  $R \rightarrow S$ .

**Remark 6.7.** If  $S \xrightarrow{\sim} T$  where  $S$  is a non-SFT subshift, then there exists a homomorphism  $f : S \rightarrow T$  which fails to be steady. This is true in particular if  $S$  is strictly sofic. To see this, suppose  $\varphi : S \xrightarrow{\sim} T$  and let  $f = \varphi|_S$ . If  $f$  were steady, then there would exist an SFT  $S'$

containing  $S$  such that  $f$  is well-defined on  $S'$  and  $f(S') = T$ . Then, for all  $r$  sufficiently large,  $f_r$  is well-defined and  $S_r \subset S'$ , so  $f_r^{-1}(T) = S_r$ . But  $S_r \neq S$  because  $S$  is non-SFT, contradicting Proposition 6.6.

Remark 6.7 points to a seeming resemblance between Proposition 6.6 and the following difficult open problem.

**Problem 6.8.** *Give necessary and sufficient conditions for a homomorphism defined on a sofic shift to be steady.*

One motivation for Problem 6.8 is to characterize the limit sets of stable cellular automaton maps. These limit sets are the subshifts  $T$  for which there exists a cellular automaton map  $f$  and a positive integer  $r$  such that  $T = \text{Image}(f^r) = \text{Image}(f^{r+1})$ . In [13], Maass studies such limit sets, and shows that they are precisely the mixing sofic shifts  $S$  which contain a receptive fixed point and which admit a steady epimorphism  $f : S \rightarrow S$ .

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