Fall 2007 Math 221 Exam 2 Review

Emily King T.A.

October 23. 2007

9.4 : Approximation of Definite Integrals

1. **Goal**: Approximate $\int_a^b f(x)dx$.

2. **Midpoint Rule**: We approximate the "area" $\int_a^b f(x)dx$ using n rectangles or equal width with heights determined by the values of f at the midpoints of the rectangle bases.

   (a) **Picture**:

   Figure 1: Midpoint Rule

   (b) **Formula**:

   $$\int_a^b f(x)dx \approx \triangle x\left[f(x_0)+f(x_1)+\ldots+f(x_{n-1})\right]$$

   where $\triangle x = \frac{b-a}{n}$.

3. **Trapezoidal Rule**: We approximate the "area" $\int_a^b f(x)dx$ using n trapezoids of equal width.

   (a) **Area of Triangle** A trapezoid with heights h_1 and h_2 and width w has area $\frac{1}{2}(h_1+h_2)w$, which is the average of the areas of the rectangles of width w and height h_1 and of width w and height h_2.

Figure 2: Trapezoid

Figure 3: Trapezoidal Rule

(b) **Picture** See figure 3.

(c) **Formula**:

$$\int_a^b f(x)dx \approx \frac{\triangle x}{2}\left[f(a_0)+2f(a_1)+\ldots+2f(a_{n-1})+f(a_n)\right]$$

where $\triangle x = \frac{b-a}{n}$.

4. **Simpson's Rule**: We approximate the "area" $\int_a^b f(x)dx$ using auxillary parabolas. Alternatively, we find a value between the approximations yielded by the Midpoint Rule and the Trapezoidal Rule.

   Formula: If M and T are the values obtained from the Midpoint Rule and the Trapezoidal Rule, respectively, for a certain n, then $S = \frac{2}{3}M + \frac{1}{3}T$, is the value obtained by Simpson's Rule for that n.

5. **Problems**

   (a) Approximate the following using the Trapezoidal, Midpoint and Simpson's Rules with $n = 5$:

   $$\int_0^2 \sqrt{1+x^3}dx$$

   (b) **Extra problems from textbook**: p 473: $7-24$

9.5: Applications of the Integral

1. **Present Value of a Continuous Income Stream**: Assume that there is a continuous income stream $K(t)$ which is deposited into a (continuously compounded) account with an annual interest rate of r. The money is deposited from time T_1

until time T_2. The value of the income stream a time $t = T_1$ is $\int_{T_1}^{T_2} K(t)e^{-rt}dt$. Typically we want to know the present value of the income stream with $T_1 = 0$.

Inspiration: If money is deposited once into an account with continuously compounded interest with a rate r and after T years the account is worth \$$A$, then the amount originally deposited is Ae^{-rt}. So the formula above is roughly adding up the current values each time money from the income stream is deposited.

2. **Total Population in a Ring Around a City Center**: If $D(t)$ is the density of the population (in people/sq. mi or some other similar unit) at a distance of t miles away from the center, then the number of people living between a miles and b miles away from the city center is approximated by $\int_a^b 2\pi t D(t)dt$.

   Inspiration: Recall that the circumference of a circle of radius t is $2\pi t$. This formula roughly says that we are adding up the population along each circle from radius a to radius b.

3. **Problems**:

   (a) (from Justin's Exam 1 Sample 2) The population density t miles from the center of a small city (in hundreds of people) per square mile is given by $D(t) = 200e^{-0.3t}dt$. How many people live in the range of 5 to 10 miles?

   (b) A continuous stream of income is produced at the rate of $10e^{1-0.03t}$ thousand dollars per year at year t, and the invested money earns 6% interest annually. What is the present value of this stream over 3 years?

   (c) **Extra problems from textbook**: pp 480–481: 1–12

9.6: Improper Integrals

1. **Goal**: We want to integrate over “infinitely long areas”.

2. **Definitions**:

(a) For some function g, $\lim_{b\to\infty} g(b)$ is the value of $g(b)$ as b takes larger and larger (positive) values and $\lim_{c\to-\infty} g(c)$ is the value of $g(c)$ as c takes larger and larger negative values.

(b) Let a be fixed. If $\lim_{b\to\infty}\int_a^b f(x)dx = L$, L finite, we define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx = L.$$

We say that the improper integral $\int_a^\infty f(x)dx$ is convergent.

(c) Let a be fixed. If $\lim_{c\to-\infty}\int_c^a f(x)dx = L$, L finite, we define

$$\int_{-\infty}^a f(x)dx = \lim_{c\to-\infty}\int_c^a f(x)dx = L.$$

We say that the improper integral $\int_{-\infty}^a f(x)dx$ is convergent.

(d) Fix some a. If $\int_a^\infty f(x)dx = L_1$ and $\int_\infty^a f(x)dx = L_2$ both converge, then we say that the improper integral $\int_{-\infty}^\infty f(x)dx$ converges and

$$\int_{-\infty}^\infty f(x)dx = L_1 + L_2.$$

3. **Problems**

(a) $\int_1^\infty e dx$

(b) $\int_{-\infty}^0 2e^{2x}dx$

(c) $\int_3^\infty \frac{1}{x\ln x}dx$

(d) $\int_{-\infty}^0 \frac{x}{(x-1)^3}dx$

(e) **Extra problems from textbook**: p 486: 21–44

10.1: Solutions of Differential Equation

1. **Checking Solutions**: If you have a differential equation, then $f(t)$ is a solution if and only if the equation remains true when y is replaced by $f(t)$, y' by $f'(t)$, y'' by $f''(t)$ and so on.

2. **Constant Solutions**: To find the constant solutions to a differential equation, set $y = C$ and $0 = y' = y' = y''' \ldots$. Whatever constants C solve the equation correspond to the constant solutions.

3. **Antiderivative Solutions**: If $y' = g(t)$, where $g(t)$ is a function of t but not y, then $y = \int g(t)dt$.

4. **Probems**

   (a) What are the constant solutions of the differential equation $y'' + ty' = y^3 - 2y^2 + y$?

   (b) (Textbook, 10.1 Ex 2) Show that the function $f(t) = \frac{1}{9}t + \sin(3t)$ is a solution of the differential equation $y'' + 9y = t$.

   (c) Solve the initial value problem $y' = 6t + 4$, $y(0) = 3$.

   (d) If the function $f(t)$ is a solution of the initial value problem $y' = \cos(t) + y^2$, $y\left(\frac{\pi}{4}\right) = 0$, find $f\left(\frac{\pi}{4}\right)$ and $f'\left(\frac{\pi}{4}\right)$.

   (e) **Extra problems from textbook**: pp 497 - 498: 1 – 12.

10.2: Separation of Variables

1. **Method** Given a differential equation of the form $y' = p(t)q(y)$, we solve the differential equation by the following steps:

$$\begin{aligned} \frac{dy}{dt} &= p(t)q(y) \\ \int \frac{1}{q(y)}dy &= \int p(t)dt \end{aligned}$$

   and integrating.

2. **Problems**: Solve the following initial value problems

   (a) $y' = \left(\frac{\sec(t)}{y}\right)^2 e^{-y^3}, y\left(\frac{\pi}{4}\right) = 0$

   (b) $\csc(t)\frac{dy}{dt} = yt, y(0) = 5$

   (c) **Extra problems from textbook**: p 506: 1–30

10.3: First Order Linear Differential Equations

1. **Method** Assume that we have a differential equation of the form $y' + a(t)y = b(t)$, where $a(t)$ and $b(t)$ are continuous on a given interval. Further let $A(t)$ be the antiderivative of $a(t)$ with no constant term. Then

$$\begin{aligned} y' + a(t)y &= b(t) \\ e^{A(t)}y' + e^{A(t)}a(t)y &= e^{A(t)}b(t) \\ \frac{d}{dt}\left(e^{A(t)}y\right) &= e^{A(t)}b(t) \\ e^{A(t)}y &= \int e^{A(t)}b(t)dt \end{aligned}$$

2. **Problems** Solve the following initial value problems

   (a) Assume $t > 0$. $ty' = t\ln t - \frac{1}{2}y$ with $y(1) = 1$.

   (b) $y' + 3y\sin(3t) = \sin(3t)$, $y\left(\frac{\pi}{2}\right) = \frac{2}{3}$.

   (c) **Extra problems from textbook**: p 514: 19–40

10.1/10.4/10.6: Applications of Differential Equations

1. **Accounts**

   (a) **Savings/Retirement Accounts**: Say someone is (continuously) depositing money into an account at a rate of \$$D(t)$ per year t. Further suppose that the account (continuously) compounds interest at an annual rate of r. Let $A(t)$ denote the amount of money in the account t years after the initial deposit. A differential equation satisfied by $y = A(t)$ is

$$y' = \underbrace{D(t)}_{\text{deposits}} + \underbrace{ry}_{\text{interest}}.$$

   If the initial deposit is M dollars, then $y(0) = M$.

(b) **Loans**: Say someone is (continuously) paying off a loan at a rate of $\$D(t)$ per year t. Further suppose that the loan (continuously) accrues interest at an annual rate of r. Let $A(t)$ denote the amount of money still owed t years after the initial deposit. A differential equation satisfied by $y = A(t)$ is

$$y' = \underbrace{-D(t)}_{\text{payments}} + \underbrace{ry}_{\text{interest}}.$$

If M dollars is originally taken out for the loan, then $y(0) = M$.

2. **Cooling/Osmosis**

(a) **Newton's Law of Cooling**: Let $f(t)$ denote the temperature of an item after t minutes (seconds, hours, etc) and $T(t)$ denote the surrounding temperature (which may remain constant over time). By Newton's Law of Cooling, the differential equation satisfied by $y = f(t)$ is $y' = k(T(t) - y)$ for some constant k.

(b) **Osmosis**: Let $f(t)$ denote the concentration of solute(s) in a solution on one side of a semi-permeable membrane after t minutes (seconds, hours, etc) and $C(t)$ denote the concentration of solute(s) on the other side (which may remain constant over time). By osmosis, the differential equation satisfied by $y = f(t)$ is $y' = k(C(t) - y)$ for some constant k.

3. **Logistic Curves**: The growth of a population $N(t)$ is often described by a logistic equation with solution $y = N(t)$:

$$y' = \frac{r}{K} y(K - y),$$

where the constant K is the carrying capacity of the environment and r is the intrinsic growth rate; that is, r is the rate of growth that would occur without environmental restrictions. A similar model determines the spread of an epidemic through a group, where $K = 100\%$ or the total size of the group, depending on the units of $N(t)$.

4. **Problems**:

   (a) Prunella is planning for her retirement. She opens a savings account with \$5000. She then deposits 2000 dollars a year into the account which earns 5% interest annually, compounded continuously. Let $A(t)$ denote the amount that Prunella has in her account at time t. Set up and solve an initial value problem for which $y = A(t)$ is a solution. How long will it take Prunella to save up $\$50,000$?

   (b) (textbook 10.4 Ex 5) Suppose that Jimmie turns on an oven in order to bake a frozen pizza. Instead of waiting for the oven to heat up, he puts the pizza in right when he turns the oven on. The temperature of the oven is given by $T(t) = 70 + 50t$ for $0 \leq t \leq 8$ minutes. Suppose further that the initial temperature of the pizza was 27° and the constant of proportionality is $k = 0.1$. Let $f(t)$ denote the temperature of the pizza. Write, but do not solve, an initial value problem for which $y = f(t)$ is the solution.

   (c) (textbook 10.6 ex 2) A pond on Fishy Joe's fish farm has a carrying capacity of 1000 fish. The pond was originally stocked with 100 fish. Let $N(t)$ denote the number of fish in the pond after t months. Set up a logistic differential equation satisfied by $N(t)$. Find the highest rate of growth, given that the intrinsic growth rate is 0.3.

   (d) **Extra problems from textbook**: p 498: 19 – 21, pp 520 – 523: 1 – 25, pp 541 – 543: 7 – 24

10.5: Graphing Solutions of Differential Equations

1. **Autonomous Differential Equations**: An autonomous differential equation is one of the form $y' = g(y)$. The solutions to autonomous differential equations have the following properties for "nice enough" g (we will only be dealing with such g).

(a) Each zero of $g(y)$ corresponds to a constant solution of a differential equation.

(b) The constant solutions divide the ty-plane into horizontal strips. Each non-constant solution lies completely in one strip.

(c) Each non-constant solution is either strictly increasing or decreasing

(d) Each non-constant solution is either asymptotic to a constant solution or else increases or decreases without bound. (wording taken from the book)

2. **Method**: We would like to sketch the graph of the solution of the differential equation $y' = g(y)$ with $y(0) = a$ for some number a. In order to do this we first plot $g(y)$ over y, as well as a blank ty-plane. For each c such that $g(c) = 0$, plot the horizontal line $y = c$ on the ty-plane. If $g(a) < 0$, then we know that our solution is decreasing through out its horizontal strip. Similarly, if $g(a) > 0$, the solution will be increasing. Finally determine the concavity of the solution and plot it. (This will be explained better in the review. Also reread your lecture notes to see more examples.)

3. **Problems**

(a) (Justin's exam 2 sample 2) Consider the autonomous differential equation $y' = -2(y+2)(y-10)$. Sketch solutions corresponding to $y(0) = c$ for $c = -5, 0, 5$, using techniques from §10.6. Indicate inflection points in your solutions if appropriate.

(b) Sketch the graphs of the solutions of $y' = 500 - 2y$ corresponding to $y(0) = 100$ and $y(0) = 300$.

(c) **Extra problems from textbook**: pp 530 – 532: 7 – 32