Fall 2007 Math 221 Exam 3 Review

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11.2 : Newton-Raphson Method

1. **Goal**: Approximate the zero of a function.

2. The Newton-Raphson method is an algorithm which is used to approximate a zero of a function. It works by finding the x-intercept of the tangent line at the previous approximation. In general, each iteration of the method yields a better approximation. In general, each iteration yields a better approximation. In order to find a zero of $f(x)$ near the point $x = x_0$ we calculate

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &\vdots \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \text{for } n \geq 0 \end{aligned}$$

3. **Problems**

   (a) Use the Newton-Raphson method to approximate an x-intercept for $f(x) = 2x^3 + x^2 + x + 1$. Use $x_0 = 0$ and

   find x_1 and x_2.

   (b) Use one iteration of the Newton-Raphson method with initial guess $x_0 = 3$ to approximate $\sqrt[3]{26}$.

(c) Use 2 repetitions of the Newton-Rhapson method to approximate the zero of $\sin(x) + \pi\cos(x) + x$ near $x_0 = 0$.

(d) **Extra problems from textbook**: p 567 (1 – 8, 14, 16, 19, 20)

11.3 Geometric Series

1. The infinite series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots$$

converges if and only if $|r| < 1$. When $|r| < 1$, the sum of the series is $\frac{a}{1-r}$. Geometric series are used to model a number of situations.

2. **Multiplier Effect**: Assume that the government cuts income taxes by $\$T$ and each person will spend MPC (MPC = Margincal Propensity to Consume) of all resulting extra income and save the rest. We would lke to estimate the total increase in spending. This is modeled by the geometric series

$$T(MPC) + T(MPC)^2 + T(MPC)^3 + \cdots$$

3. **Drug Therapy**: Assume that the body eliminates a certain drug proportionately to the amount present in the body, say 10% is eliminates each day. If someone takes a daily maintenence dose, of say 0.1 mg, then how much of the drug is in the system after an extended period of time? We estimate this using the geometric series

$$0.1 + 0.90(0.1) + 0.90(0.1)^2 + \cdots .$$

That is, if there are a_n mg of the dose in the body n days after the initial dose, then immediately after taking the dose on the $(n+1)^{st}$ day, there will be $a_{n+1} = 0.1 + 0.90a_n$ mg.

4. **Perpetuity** : A *perpetuity* is a periodic sequence of payments that continue indefinitely. The *capital value* of the perpetuity is the sum of the present values of all future paymnets. If a perpetuity promises to pay P dollars at the end of each year and the (annual) interest rate is r, compounded annually, then the present value of P dollars in k years is $P(1+r)^{-k}$. So the capital value of the perpetuity is $\sum_{k=1}^{\infty} P(1+r)^{-k}$. If the perpetuity pays at the beginning of each year, then the capital value would be $\sum_{k=0}^{\infty} P(1+r)^{-k}$.

5. **Problems**

   (a) Determine the sums of the following geometric series, when they are convergent.

      i. (Book p 576 #10) $6 - 1.2 + 0.24 - 0.048 + 0.0096 - \cdots$

      ii. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{2k+3}$

      iii. $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{-2k+3}$

   (b) Sum an appropriate infinite series to fine the rational number whose decimal expansion is $2.006\overline{767}$.

   (c) Estimate the total new spending created by a \$4 billion federal income tax cut when the population's marginal propensity to consume is 90%.

   (d) (Book p 577 #32) A patient receives M milligrams of a certain drug daily. Each day the body eliminates a fraction of q of the amount of the drug present in the system. Estimate the total amount of the drug that should be present after extended treatment, immediately after a dose is given.

   (e) Emilio Rex decides to start a scholarship fund for talented Math 221 students. The scholarship will be funded by a perpetuity in an account with 4% interest compounded annually. If one \$1000 scholarship will be paid at the end of each year, how much must initially be deposited into the perpetuity?

(f) **Extra problems from textbook**: pp 576 – 577 (1 – 21, 23 – 32, 35 – 40)

11.1 / 11.5 Taylor Polynomials / Taylor Series

1. Given a function $f(x)$, the *nth Taylor Polynomial* of $f(x)$ at $x = a$ is the polynomial defined by

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

where $k! = k(k-1)(k-2)\cdots 2\cdot 1$. This polynomial has the same value and derivatives, up to order n, at $x = a$ as f does. When $a = 0$, the formula is

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

Further notice that $p_1(x)$ is just the tangent line of f at $x = a$. If we continue this process indefinitely, we obtain the *Taylor series* of $f(x)$ at $x = 0$,

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

Although we won't be dealing with issues of convergence, it should be noted that the equality above may only be true for x that are sufficiently close to 0. Recall the following Taylor series about $x = 0$:

$$
\begin{aligned}
e^x &= 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots + \tfrac{1}{n!}x^n \\
&\quad \text{If } f(x) = e^x \text{, then } f^{(n)}(0) = e^0 = 1 \text{ for all } n \\
\tfrac{1}{1-x} &= 1 + x + x^2 + \cdots + x^n + \cdots \text{ for } |x| < 1 \\
&\quad \text{This is simply a geometric series.} \\
\cos(x) &= 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 - \cdots + \tfrac{(-1)^n}{(2n)!}x^{2n} + \cdots \\
&\quad \text{Consider the trig derivs at 0.} \\
\sin(x) &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots + \tfrac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots
\end{aligned}
$$

2. **Problems**:

   (a) Determine the 0^{th}, 1^{st}, 2^{nd} and 3^{rd} Taylor polynomial about $x = 0$ for $f(x) = 3x^2 + 4x + 1$. Determine the Taylor series of $f(x)$ about $x = 0$.

   (b) Determine the 1^{st} Taylor polynomial about $x = 1$ of $f(x) = x$.

   (c) Use suitable operations on the Taylor series at $x = 0$ of $\frac{1}{1-x}$, e^x, $\cos(x)$ to solve the following problems:

      i. A. Find the first 4 non-zero terms of the Taylor series at $x = 0$ of $\frac{x}{(1+x^2)^3}$.

         B. If $f(x) = \frac{x}{(1+x^2)^3}$, what are $f^{(5)}(0)$ and $f^{(6)}(0)$?

         C. What is the 7^{th} Taylor polynomial about $x = 0$ of $\frac{x}{(1+x^2)^3}$?

      ii. Find the Taylor series of $\ln(1-x) + 1$ and $\ln\left(\frac{1+x}{1-x}\right)$ at $x = 0$.

      iii. Find an infinite series that converges to the value of the definite integral $\int_0^1 x e^{x^3} dx$.

   (d) **Extra problems from textbook**: p 560 (1 – 10, 13 – 26), p 590 – 591 (1 – 25, 28, 32 – 40)

11.4 Integral Test

1. Let $f(x)$ be continuous, decreasing and positive for $x \geq 1$. Then the infinite series $\sum_{k=1}^{\infty} f(k)$ is convergent if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. Conversely, the infinite series is divergent if the improper intergral is divergent. Similarly for any integer N, the convergence of $\sum_{k=N}^{\infty} f(k)$ is completely determined by the converengence of $\int_N^{\infty} f(x)dx$. Note that in general $\sum_{k=1}^{\infty} f(k) \neq \int_1^{\infty} f(x)dx$. For example, $\int_1^{\infty} \frac{1}{x^2}dx = 1$, while $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. (The proof of this is beyond the scope of this class.)

2. **Probems**

   (a) Use the intergral test to determin whether the infinite series is convergent of divergent. Assume that the hypotheses of the integral test are satisfied.

      i. $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k\sqrt{\ln k}}$

      ii. $\displaystyle\sum_{k=2}^{\infty} \frac{e^{1/k}}{k^2}$.

   (b) Is the improper integral $\int_1^{\infty} \frac{1}{\pi^x}dx$ convergent or divergent?

   (c) For what integers p is $\sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^p$ convergent?

   (d) **Extra problems from textbook**: p 583 (1 – 16, 20)

12.1 Discrete Random Variables

1. Let X be a discrete random variable; that is, a variable that depends on chance. Then the *expected value* (or mean) of X is

$$E(X) = a_1p_1 + a_2p_2 + \cdots + a_np_n,$$

   where the a_i are the possible outcomes with the respective probabilities p_i. The *variance* describes how much the data spreads away from the mean.

It is defined by

$$\text{Var}(X) = (a_1 - E(X))^2 p_1 + (a_2 - E(X))^2 p_2 + \cdots + (a_n - E(X))^2 p_n.$$

Finally the *standard deviation* of X is

$$\sigma(X) = \sqrt{\text{Var}(X)}.$$

2. **Problems**:

   (a) One possible bet in roulette is to wager \$1 on "red." The two possible outcomes are "Lose \$1" and "Win \$1." A roulette wheel in Las Vegas has 18 red numbers, 18 black numbers and 2 green numbers. Compute the expected value, the variance and the standard deviation of the amount won.

   (b) (Book p 601 Ex 2) The production department at a radio factory sends CB radios to the inspection department in lots of 100. There an inspector examines three radios at random from each lot. If at least one of the three radios is defective and needs adjustment, the entire lot is sent back to the production department. Records of the inspection department show that the number X of defective radios in a sample of three radios has the following probability table:

	Quality Control Data			
Defectives	0	1	2	3
Probability	0.7265	0.2477	0.0251	0.0007

   i. What percentage of the lots does the inspection department reject?

   ii. Find the mean number of defective radios in the samples of three radios.

   iii. Estimate the average number of defective radios in each lot of 100 radios.

   (c) **Extra problems from textbook**: pp 601 – 602 (1 – 10)

12.2/12.3 Continuous Random Variables with Expected Value and Variance

1. Let X be a *continuous random variable* with possible values $A \leq x \leq B$. A function $f(x)$ is called a *probability density function* if for all a and b in the range of possible value of possible values of X, $\Pr(a \leq X \leq b) = \int_a^b f(x)dx$, such a function must satisfy the following properties

   - (No negative probabilies probabilities) $f(x) \geq 0$ for $A \leq x \leq B$.
   - (The sum of the probabilities is 1) $\int_A^B f(x)dx = 1$.

   Note that $\Pr(X = a) = 0$ for any a. When calulating multiple probabilities, it is easier to first compute the *cumulative distribution function*, $F(x) = \int_A^x f(t)dt$. F is an anti-derivative of f. Then

$$\begin{aligned}\Pr(a \leq X \leq b) &= \int_a^b f(x)dx \\ &= F(x)|_a^b \\ &= F(b) - F(a).\end{aligned}$$

   Note that by the properties of the PDF,

   - $F(x) \geq 0$ for $A \leq x \leq B$.
   - $F(A) = 0$ and $F(B) = 1$.

   Finally, since the CDF is an anti-derivative of the PDF, $F'(x) = f(x)$ for $A \leq x \leq B$. Analogous to the discrete case, we have the following formulas:

$$\begin{aligned}E(X) &= \int_A^B xf(x)dx \\ \text{Var}(X) &= \int_A^B (x - E(x))^2 f(x)dx \\ &= \int_A^B x^2 f(x)dx - E(X)^2\end{aligned}$$

2. **Problems**

   (a) (Book p 607 Ex 5 & p 613 Ex 3) Let X be the random variable associated with the experiment that consists of selecting a point at random from a circle of radius 1 and observing its observing its distance from the center.

      i. Find the probability density function $f(x)$ and the cumulative density function $F(x)$ of X.

      ii. Find the expected value, variance and standard deviation.

   (b) (Modified from the book p 609 #20) Suppose that at a certain supermarket the amount of time one must wait at the express lane is a random variable with density function $f(x) = \frac{11}{10(x+1)^2}$, $0 \le x \le 10$.

      i. Verify that f is a probability density function.

      ii. Find the cumulative density function F.

      iii. Find the probability of having to wait less than 4 minutes at the express lane. Find the probability of having to wait between 4 and 5 minutes.

      iv. How long should you expect to wait?

   (c) Find the value of k that makes $3kx(x + 4k)$ a probability density function for $0 \le x \le 1$.

   (d) **Extra problems from textbook**: pp 609 – 610 (1 – 12, 15 – 26, 31 – 36, 37, 38), pp 615 – 616 (1 – 24)

12.4 Exponential Random Variables

1. Let k be a positive constant. Then the function $f(x) = ke^{-kx}$, $x \ge 0$, is an *exponential desnity function.* A random variable X with an exponential density function is called an *exponential random variable.* For an

exponential random variable X,

$$E(X) = \frac{1}{k} \text{ and } \mathrm{Var}(X) = \frac{1}{k^2}.$$

2. **Problems**:

   (a) (Modified from book p 619 Ex 1) Suppose that the number of days of continuous use provided by a certain brand of light bulb is an exponential random variable with expected value 100 days.

      i. Find the density function of X.

      ii. Find the probability that a randomly chosen bulb will last between 80 and 90 days.

      iii. Find $\mathrm{Var}(X)$.

   (b) **Extra problems from textbook**: pp 624 – 625 (1 – 14)