Project List 1

1. **(Epidemiology)** The spread of a communicable disease among a population is often modeled using differential equations. To simplify the model, let's assume that the disease is not fatal and that the total population is fixed.

   (a) Assume that there is a constant "recovery rate" among the infected population, meaning that the number of infected people becoming cured per unit time is proportional to the total number of infected people. Assume that once cured, people are susceptible to being infected again, and that the "infection rate" (the chances of an uninfected person becoming infected per unit time) is proportional to the number of infected people. Construct a model for the evolution of the number of infected people over time, and discuss the underlying assumptions (in terms of the disease and of human behavior) on which this model is based.

   (b) Researchers have obtained the following estimates for the size of the infected population: 10 years ago, 0.2% of the total population; 5 years ago, 0.8%; and now, 3%. What parameters in your model are consistent with these observations? Based on these parameters, how will the size of the infected population evolve in the future? How would your predictions change if the current percentage of people that are infected were instead 2.9%, or 3.1%? Also, what would you do if more or fewer data points were known?

   (c) Assume now that a certain fraction of the population being cured at a given time becomes immune to the disease, and that people who are cured without becoming immune still have the same chance of becoming immune after being reinfected and cured again. How can the model be modified to take immunity into account? Is the new model consistent with the data above, and if so is any additional data needed to predict how the size of the infected population will evolve? What will happen in the long run?

   (d) How could you make the model more realistic by taking into account changes in the population due to factors like births, deaths and immigration/emigration? Qualitatively speaking, how would these factors affect your results?

2. **(Cooling)** You are called to a murder scene, and are asked to determine the time of the murder based the temperature of the victim's body. Assume that the body starts to cool immediately at the time of death according to Newton's law of cooling — the rate of cooling is proportional to the difference between the body's temperature and the ambient temperature. Normal human body temperature is about 37°C.

   (a) Suppose that the body is discovered in a room whose temperature is assumed to have been constant at 21°C. At the time you reach the scene, the body has a temperature of 24°C. How will you determine the time of death, and how will you explain your analysis to a jury?

   (b) Suppose that in fact the room temperature is found to fluctuate between 20°C to 22°C, but that all other measurements are very accurate. What bounds can you put on the error in your estimate of the time of death? How would the size of the error differ if the body had a different temperature at the time of discovery?

   (c) What if the fluctuations between 20°C and 22°C are caused by a thermostat and are periodic in time? Incorporate this effect into your model and discuss the extent to which you can narrow the error bounds on the time of death as a result.

   (d) In real life the cooling of a body does not follow Newton's law of cooling too closely, due to a number of factors. Among these are that some metabolic processes continue (generating internal heat) for a while after death, and that the temperature of the body at a given time is not uniform

— it will cool from the outside in. How can you make your model more realistic and how would these factors affect your results?

3. **(Pollution)** A factory has been releasing a pollutant chemical into a lake at a constant rate for a long time. The factory is the sole source of this chemical in the lake. Government inspectors measure the pollutant concentration in the lake, and inform the company that owns the factory that the concentration must be reduced to at most 30% of its present value by the time of a follow-up inspection one year from now. The lake has a volume of $110,000,000$ cubic meters, and water flows into and out of the lake at an average rate of 7.4 cubic meters per second. The company decides that it is economically infeasible to defy the government or to modify its factory to produce less of the chemical per unit time, and instead decides to shut down the factory for part of the year in order to pass the inspection.

   (a) Construct a model that expresses how the pollutant concentration varies over time, both when the factory is operating and when it isn't. What assumptions go into this model?

   (b) If the factory continues operating for the first part of the year and then shuts down for the rest of the year, how long can it operate without failing the year-end inspection?

   (c) What if the factory shuts down immediately and then resumes operation later in the year — can it operate for a longer period with this strategy, or with some other strategy?

   (d) After the first inspection is passed, the inspectors will return yearly to check that the pollutant concentration remains below the required level. For what fraction of each year can the factory operate? What will the maximum level of pollutant be during the year?

   (e) Discuss how your results would differ for different lakes, and how you could take into account other factors such as nonconstant rates of flow into and out of the lake.

4. **(Preventive Maintenance)** How frequently should preventive maintenance be performed? Here is a specific scenario to model. A company operates a machine for which a key part tends to wear out relatively frequently. Inspecting the part does not seem to help predict when it will fail, but past observations have suggested that a part that has been working for $n-1$ days will fail on the nth day with probability $1 - p^n$, where $p = 0.998$.

   (a) How many days will such a part operate before failing, on average?

   (b) Replacing a working part with a new one costs $200, while if the part fails during operation, damage to the rest of the machine generally occurs, so that the average cost of repair and replacement is $1000. Should the part be replaced automatically after a certain number of days of use? If so, how many days is best and how much money will the company save compared to the strategy of only replacing the part when it fails?

   (c) Suppose the cost of replacement before failure is X and the cost of replacement after failure is Y, where $Y > X$. Are there some values of X and Y for which preventive maintenance after a certain number of days is worthwhile and other values for which it isn't? Discuss in particular what your recommendations would be for cases when X is very small compared with Y, and when X is close to Y.

   (d) What if the probability of a part that has been working for $n-1$ days failing on the nth day were given by a different formula or simply a table of numbers — how then would you decide at what interval do preventive maintenance?

   (e) What other factors might come into play in a real-life situation, and how can your model be changed to incorporate them?