Eric Slud 5/4/07

Stat 650 Sample Final Problems

Instructions. These problems are intended to be roughly of the difficulty and topic coverage as the problems you will be given on the Final Exam on Wednesday, May 16, 2006. The main topic and idea for each problem is indicated in italics. The Final Exam will consist of about 5 or 6 problems like the single-topic problems, and I may again include a short-answer (true/false style) section.

The first five problems relate to the `M/M/1` queue with arrival rate $\lambda = 1$ and service rate $\mu = 3$.

(1). (*Recursive equation based on conditioning and unconditiing.*) Find $E_0(V_2)$ (the expected time for an empty queue until the first time there are 2 customers in the system).

(2). (*Martingale calculation of hitting probabilities*) Find $P_2(\text{hit 5 before 0})$ and $E_2(\text{time to hit } \{0, 5\})$. *Hint: before hitting 0, the Markov Chain X_t is just a random walk in continuous time. You will need two different martingales in terms of X_t to solve this problem.*

(3). (*Cycles in continuous-time Markov Chains*) Find the long-term proportion of time spent in state 3 immediately following a transition from state 2.

(4). (*Cycles in discrete-time Markov Chains*) Find the long-term proportion of transition steps spent in an even-numbered state.

(5). (*Conditioning*) Suppose that the owner of the M/M/1 service facility experiences a cost equal to the person-hours which customers have waited before their service begins. If the initial state of the queue is $X(0) = 2$ (one customer on line and one being served), then find the mean and variance of the owner's accumulated cost up to the time of the first service completion.

(6). (*Stationary distributions of chains with transient states*) The continuous-time Markov chain X_t with time-t transition matrix $P(t)$ has state-space $S = \{A, B, 1, 2, 3\}$ and intensity-matrix

$$Q = \begin{pmatrix} -1.5 & 1 & .5 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 & 1 \\ 0 & 0 & 2 & -5 & 3 \\ 0 & 0 & 1 & 3 & -4 \end{pmatrix}$$

(a). Find the transition matrix P^* of the embedded discrete-time Markov-chain associated with Q.

(b). Find $\lim_{n\to\infty} (P^*)^n$ and $\lim_{t\to\infty} P_{B1}(t)$.

(7). (*Exponential r.v.'s and Poisson process*) If $N_1(t)$ and $N_2(t)$ are independent Poisson processes with respective rates 3 and 1, then find $P(N_1(t) \text{ hits } 6 \text{ before } N_2(t) \text{ hits } 3)$ numerically.

(8). (*Exponential r.v.'s*) Let $T_1, \ldots, T_5$ be independent random variables, $T_k \sim \text{Expon}(k)$. Find $P(\max(T_4, T_5) < \min(T_1, T_2, T_3))$. (*Hint: consider superposed process.*)

(9). (*Martingales and hitting probabilities – discrete & continuous-time processes; compound Poisson or branching processes*) Suppose that ξ_j for $j \ge 1$ are *iid* random variables such that ξ_j falls with equal probabilities $1/3$ on each of the values $-1, 0, 1$, and suppose $N(t)$ is an independent Poisson process with rate 2. Define $Y_n = \sum_{j=1}^n \xi_j$, $X(t) = Y_{N(t)}$.

(a) Show that Y_n is a martingale with respect to $\{\xi_1, \ldots, \xi_n\}$.

(b) Show that $X(t)$ is a martingale in the sense that for each $s < t$,

$$E\Big(X(t) - X(s) \,|\, (N(u),\, u \le s),\ (\xi_j,\ j \le N(s)) \Big) = 0$$

and also that $X^2(t) - 4t/3$ is a martingale in the same sense.

(c) Find $P(X(t) \text{ hits -10 before } 15)$ and

$$E\Big(\inf\{t > 0 : \ X(t) \in \{-10, 15\}\}\Big)$$

(10). (*Poisson process & Markov Chain*) Suppose that $X_k(t)$ for $k = 1, 2, \ldots$ is an independent sequence of Markov chains on a finite state-space S, all starting with the same initial distribution μ (a probability vector indexed by S), and let $N(t)$ be a Poisson unit-rate process independent of all of the $X_k(\cdot)$ processes. Show that for each t, the variables

$$R_j(t) = \sum_{k=1}^{N(t)} I_{[X_k(t)=j]} \quad , \qquad j \in S$$

are independent Poisson random variables, and find $\lim_{t\to\infty} R_j(t)/t$.

(11). (*Reversibility & detailed balance*) Show that every two-state irreducible continuous-time Markov chain satisfies the detailed balance condition.

(12). (*Formulation of intensity matrix and MC stationary distributions*) Suppose that a continuous-time system is defined with states $\{0, 1, 2, 3\}$ in terms of two independent Poisson processes $N_j(t)$ with respective rates $1, 2$ by $X(t) = 3N_1(t) - N_2(t) \mod 4$. Show that the system is Markovian, and find its intensity matrix and its stationary probability of being in state 3.

(13). (*Recurrence criterion*) Suppose that $(X_k,\ k \geq 0)$ is an irreducible discrete-time homogeneous Markov chain with the nonnegative integers as states such that, for all sufficiently large i, $\sum_{j\in S} P_{ij}(j-i) < 0$.

(a) Show that the transition matrix P can be modified on finitely many rows to give another irreducible transition matrix P^* with the property that the HMC governed by P^* is recurrent.

(b) Conclude from (a) that $\{X_k\}_k$ is also recurrent.