Stat 700 Take-Home Test

Instructions. You are to work on this take-home test individually. You may get information from books and notes and you may get hints from me, but that is all. A perfect score is 100 points, but 120 points are possible. The test is due either at my office or my mailbox by 4pm, Friday Dec. 12.

(I) (*35 points*) Consider the dataset $\{(X_i,\, Y_i)\, :\; i = 1, \ldots n\}$ of i.i.d. random pairs, assumed to have distributions satisfying

$$X_i \sim \mathcal{N}(\mu, \sigma^2) \quad , \qquad Y_i - \rho X_i \sim \mathcal{N}(\nu, \frac{1}{2}\sigma^2)$$

where X_i and $Y_i - \rho X_i$ are independent for each i, and the parameter-vector $\vartheta = (\mu,\, \nu,\, \rho,\, \sigma^2) \in \mathbf{R}^3 \times \mathbf{R}^+$ is unknown.

(a). Show that these data have an exponential family joint density which is not a natural canonical family but is of rank greater than the parameter dimension with identifiable parameter, and find (and justify that you have found) the minimal sufficient statistic vector $T(\mathbf{X}, \mathbf{Y})$ for ϑ. (*For this, it is enough to show for your sufficient statistic that the density ratio* $f(\mathbf{x}, \mathbf{y}, \vartheta)/f(\mathbf{x}, \mathbf{y}, \vartheta_0)$ *for a fixed parameter value* ϑ_0 *as a function of the parameter uniquely determines your* $T(\mathbf{x}, \mathbf{y})$.)

(b). Find the maximum likelihood estimator of the parameter-vector ϑ.

(II). (*45 points*) Let V_i for $i = 1, \ldots, n,$ be a sample of *Expon*(λ) variables, and let the discrete random variables U_i be defined from V_i as $U_i = [V_i]$ equal to the greatest integer less than or equal to V_i, i.e., $U_i = k$ whenever $k \le V_i < k+1$, for $k \ge 0$.

(a). Find the exact Cramer-Rao lower bounds for the variance of an unbiased estimator of $e^{-\lambda}$ based respectively on samples $\mathbf{V} = \{V_i\}_{i=1}^n$ and on $\mathbf{U} = \{U_i\}_{i=1}^n$.

(b). Show, giving full details in support of your argument, that there is a UMVUE for $e^{-\lambda}$ based on $\mathbf{V}$ and *also* a UMVUE for $e^{-\lambda}$ based on $\mathbf{U}$. (*You do not necessarily have to find these estimators explicitly to do this or the next part of the problem.*)

(c). Does either of the UMVUE's in (b) attain the corresponding Cramer-Rao lower bound in (a) for every value of λ ? Give as definitive an answer as you can in light of the result in Theorem 3.4.2 of Bickel and Doksum.

(III). (*40 points*) A sample of 10 independent variables X_i, $i = 1, \ldots, 10$, is observed, with values in $(0,1)$ and density known to have the form

$$f_X(x, \vartheta) \;=\; \begin{cases} \vartheta & \text{if} \quad -1 \,<\, x \,<\, 0 \\ 2\,(1-\vartheta)\,x & \text{if} \quad 0 \,\le\, x \,<\, 1 \end{cases}$$

(a). Find a most powerful test φ of the null hypothesis $H_0 : \vartheta = 0.4$ versus the alternative hypothesis $H_1 : \vartheta = 0.75$ of size 0.05. What is the size of a most powerful level .05 test φ_0 *which involves no auxiliary randomization* (i.e., is a nonrandomized test of the same hypotheses) ? Find the power of both φ and φ_0 against the alternative $\vartheta = 0.75$.

(b). Are either or both of the same tests φ, φ_0 uniformly most powerful (UMP) in testing H_0 against the more general one-sided alternative $H_A : \vartheta \ge 0.5$? uniformly over which set of competing tests ?

(c). For the parameter space $\Theta = (0,1)$, action space $\mathcal{A} = \{0,1\}$, loss function $L(\vartheta, a) = 2a\, I_{[\vartheta \le 0.4]} \,+\, (1-a)\, I_{[\vartheta > 0.4]}$, and prior $\pi \sim \text{Unif}(0,1)$, what is the optimal Bayes decision rule ?