Solutions to Stat 701 Take-Home Test, Spring 2009

(I) (a) The estimators $(\hat\alpha,\, \hat\beta)$ minimizing $\sum_{i=1}^n (Y_i - \alpha - \beta Z_i)^2$ satisfy

$$\sqrt{n} \cdot \begin{pmatrix} \hat\alpha - \alpha \\ \hat\beta - \beta \end{pmatrix} = \begin{pmatrix} 1 & \bar Z \\ \bar Z & n^{-1}\sum_i Z_i^2 \end{pmatrix}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_i \begin{pmatrix} 1 \\ Z_i \end{pmatrix} \qquad (1)$$

$$= \frac{1}{(n-1)\, s_Z^2} \begin{pmatrix} n^{-1}\sum_i Z_i^2 & -\bar Z \\ -\bar Z & 1 \end{pmatrix} \frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_i \begin{pmatrix} 1 \\ Z_i \end{pmatrix}$$

where $\epsilon_i = Y_i - \alpha - \beta Z_i$ for the true (α, β). Note that by the Law of Large Numbers, as $n \to \infty$,

$$\begin{pmatrix} \hat\alpha - \alpha \\ \hat\beta - \beta \end{pmatrix} = \begin{pmatrix} 1 & \bar Z \\ \bar Z & n^{-1}\sum_i Z_i^2 \end{pmatrix}^{-1} \xrightarrow{P} \begin{pmatrix} 2 & -\lambda \\ -\lambda & \lambda^2 \end{pmatrix}$$

By the multivariate CLT, the limiting distribution of (1) is normal with mean 0 and variance (under the true model)

$$\begin{pmatrix} 2 & -\lambda \\ -\lambda & \lambda^2 \end{pmatrix} E(\begin{pmatrix} t \\ Z_1 \end{pmatrix}^{\otimes 2} Z_1 \sigma^2) \begin{pmatrix} 2 & -\lambda \\ -\lambda & \lambda^2 \end{pmatrix} = \sigma^2 \begin{pmatrix} 2/\lambda & -2 \\ -2 & 3\lambda \end{pmatrix}$$

(b) This matrix V can be estimated consistently by plugging into the expression for V the estimators $\hat\lambda = 1/\bar Z$, $\hat\sigma^2 = n^{-1}\sum_{i=1}^n (Y_i - \hat\alpha - \hat\beta Z_i)^2$, or more directly by

$$\begin{pmatrix} 1 & \bar Z \\ \bar Z & n^{-1}\sum_i Z_i^2 \end{pmatrix}^{-1} \frac{1}{n} \sum_{i=1}^n \begin{pmatrix} 1 \\ Z_i \end{pmatrix}^{\otimes 2} (Y_i - \hat\alpha - \hat\beta Z_i)^2 \begin{pmatrix} 1 & \bar Z \\ \bar Z & n^{-1}\sum_i Z_i^2 \end{pmatrix}^{-1}$$

(II) (a) Using $a(s) \equiv 1$, we get the Rao Score statistic, after noting $\hat\lambda^{(r)} = 1/\bar Y$, by

$$\frac{\partial}{\partial \vartheta} \sum_{i=1}^m \log(\frac{1}{2}\, \lambda \exp(\vartheta X_i \, - \, Y_i\, \lambda\, e^{\vartheta X_i}) \Big|_{\vartheta=0,\, \lambda = \hat\lambda^{(r)}} = \sum_{i=1}^n X_i\, (1 - Y_i/\bar Y)$$

The statistic $S_n = -(\sqrt{n}\, \bar Y)^{-1} \sum_{i=1}^n X_i(Y_i - \bar Y)$ is the same under H_0 as $(-\lambda/\sqrt{n}) \sum_{i=1}^n X_i(Y_i - \bar Y)$ which has variance $1/4$. The test rejects when $S_n \ge 0.5\, z_\alpha$.

(b). At a fixed alternative $\vartheta > 0$, note $E(Y_i|\, X_i = 0) = 1/\lambda$ and

$$E(Y_i\,|\,X_i = 1) \;=\; \int_0^\infty \exp(-\lambda \int_0^t e^{\vartheta\, a(s)}\, ds)\, dt \;\equiv\; \frac{\gamma}{\lambda} \;<\; \frac{1}{\lambda}$$

Then $S_n/n \;\rightarrow\; (\gamma - 1)/(\gamma + 1)$, which implies consistency in the form $P(S_n/\sqrt{n} \geq 0.5\, z_\alpha) \;\rightarrow\; 1$.

(c). With $a(s) \;=\; 1 - 0.5\, I_{[s>2]}$, check that γ defined in (b) has the form $\gamma \;=\; 1 \;-\; (1 - 0.5e^{-2\lambda})/\sqrt{n} \,+\, o(1/\sqrt{n})$. Then using what we found in class, that (under regularity conditions) the asymptotic variance of statistics under contiguous alternatives are the same as under the null hypothesis, we find under the model with $\vartheta = 1/\sqrt{n}$,

$$\frac{1}{\sqrt{n}}\, S_n \;\xrightarrow{\mathcal{D}}\; \mathcal{N}(\frac{1}{2} \;-\; \frac{1}{4}\, e^{-2\lambda},\; \frac{1}{4})$$

So the power is $\Phi(1 \;-\; \frac{1}{2}\, e^{-2\lambda} \;-\; z_\alpha)$.

(III) Let $F(x) = e^x/(1 + e^x)$ be the logistic distribution function and $f = F'$ the density, and let $\vartheta = (\mu, \sigma)$ be the parameter vector. Then after calculating $\nabla^{\otimes 2}\, \log(\sigma^{-1}\, f((x - \mu)/\sigma))$, the information matrix is found to have the form

$$I_X(\vartheta) \;=\; \frac{1}{\sigma^2} \begin{pmatrix} 2\int f^2 dx & 0 \\ 0 & -1 + 4\int xF(x)f(x)dx \end{pmatrix} \;=\; \frac{1}{\sigma^2} \begin{pmatrix} 1/3 & 0 \\ 0 & 1.42996 \end{pmatrix}$$

and the inverse is the asymptotic MLE variance. The corresponding information for the multinomial setting is

$$I_N(\vartheta) \;=\; -\sum_{j=1}^{4} p_j(\vartheta)\, \nabla^{\otimes 2}\, \log p_j(\vartheta) \quad , \quad p_j(\vartheta) \;=\; F(\frac{a_j - \mu}{\sigma}) - F(\frac{a_{j-1} - \mu}{\sigma})$$

We invert both matrices to find the respective asymptotic variances of $\sqrt{n}(\hat{\vartheta} - \vartheta)$ and $\sqrt{n}(\tilde{\vartheta} - \vartheta)$.

(b) There is actually a derivation in Bickel-Doksum (p. 402) that this statistic is the Rao Score Test statistic in the multinomial-data setting with logistic distribution interval-probabilities $p_j(\vartheta)$. Alternatively, we showed by a small Taylor series argument in class how the Likelihood ratio test statisic was asymptotically equivalent to this Pearson statistic, so the Wilks Theorem gives the asymptotic null distribution as $\chi^2_{4-1-2} = \chi^2_1$.

(c) The main hint was given in class:

$$\frac{N_1 - n\,F((-1-\hat{\mu}/\hat{\sigma})}{\sqrt{n}} \;=\; T_1 - T_2 \,+\, o_P(1)$$

with

$$T_1 \;=\; \frac{N_1 - np_1(\vartheta)}{\sqrt{n}} \;=\; \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(I_{[X_i\le -1]} - F(\frac{-1-\mu}{\sigma})\Big) \;\approx\; \mathcal{N}(0, p_1(\vartheta)(1-p_1(\vartheta)))$$

$$T_2 \;=\; \sqrt{n}\,(p_1(\hat{\vartheta}) - p_1(\vartheta)) \;\approx\; f(\frac{-1-\mu}{\sigma})\,\frac{1}{\sigma}\begin{pmatrix} 2F((-1-\mu)/\sigma) - 1 \\ 1 + \frac{-1-\mu}{\sigma}\,(1 - 2F((-1-\mu)/\sigma)) \end{pmatrix}'$$

$$\cdot\,\sigma^2 \begin{pmatrix} 3 & 0 \\ 0 & 1/1.42996 \end{pmatrix} \frac{\sigma^{-1}}{\sqrt{n}} \sum_{i=1}^{n} \begin{pmatrix} 2F((X_i-\mu)/\sigma) - 1 \\ 1 + \frac{X_i-\mu}{\sigma}\,(1 - 2F((X_i-\mu)/\sigma)) \end{pmatrix}$$

The variance of T_2 and covariance with T_1 are readily evaluated because both are expressed as sums of *iid* variables.