3/24/03

Selected (Theoretical) Solutions from HW 3.

Ch. 14 #4. It is only a little harder to justify the solution rigorously than to do a formal and heuristic argument, as almost all of you did. The idea is certainly to do a Taylor's expansion of $\log(Y+c)$ around the point μ, but at least a little care is required because Y is a random variable which has positive probability of making $(Y+c-\mu)/\mu$ fall outside the radius of convergence, which is 1.

On the event $\Big[|Y+c-\mu| \le \mu/2\Big]$, we can do the expansion

$$|\log(Y+c) - \log(\mu) - \frac{Y+c-\mu}{\mu} + \frac{1}{2}\left(\frac{Y+c-\mu}{\mu}\right)^2 - \frac{1}{3}\left(\frac{Y+c-\mu}{\mu}\right)^3|$$

$$\le \sum_{j=4}^{\infty} \frac{1}{j\mu^j}\,|Y+c-\mu|^j \;\le\; \frac{(Y+c-\mu)^4}{2\mu^4}$$

Next, using the fact that $E(Y-\mu)^j = \lambda$ for $j = 2, 3$, we find

$$E\Big(\frac{Y+c-\mu}{\mu} - \frac{1}{2}\left(\frac{Y+c-\mu}{\mu}\right)^2 - \frac{1}{3}\left(\frac{Y+c-\mu}{\mu}\right)^3\Big) \;=\; \frac{c-1/2}{\mu} + \mathcal{O}(\mu^{-2})$$

Finally, a remainder estimate using the exponentially small (in λ) probability of $|Y+c-\mu| \le \mu/2$ in order to obtain

$$E\Big\{(\log(Y+c) \,+\, \log(\mu) \,+\, \frac{Y}{\mu} \,+\, \frac{Y^2}{2\mu^2} \,-\, \frac{Y^3}{3\mu^3})\; I_{[|Y+c-\mu|\ge\mu/2]}\Big\} \;=\; \mathcal{O}(\mu^{-2})$$

Nothing less than an argument of this sort is fully convincing !

Ch. 3 #34. No one did the part of this problem having to do with large-sample distibutions of the Cressie-Read power-divergence statistics. It turns out that the Poisson sampling is largely irrelevant. Under the null-hypothesis (assumed valid) that all of the true parameters π_i are the elements of a d-dimensional parameterized family $\pi(\vartheta)$, we have via standard MLE theory

$$\sqrt{n}(\hat{\vartheta} - \vartheta) \;\approx\; \left(\sum_{i=1}^{K} \frac{1}{\pi_i}(\nabla_\vartheta\, \pi_i)^2\right)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{K} (\nabla_\vartheta\, \log \pi_i)\,(n_i - n\pi_i)$$

is asymptotically normally distributed. It follows that the Cressie-Read statistics, with $\hat{\mu}_i$ replaced by $(\sum_{j=1}^{K} n_j)\pi_i(\hat{\vartheta})$, are

$$\frac{2}{\lambda(\lambda+1)} \sum_{i=1}^{K} n_i \Big((1 + \frac{\hat{\mu}_i - n_i}{n_i})^{-\lambda} - 1\Big) \approx$$

$$\frac{2}{\lambda(\lambda+1)} \sum_{i=1}^{K} n_i \Big((-\lambda)\frac{\hat{\mu}_i - n_i}{n_i} + \frac{(-\lambda)(-\lambda-1)}{2} \frac{(\hat{\mu}_i - n_i)^2}{n_i^2}\Big)$$

$$\approx \sum_{i=1}^{K} \frac{(n_i - \hat{\mu}_i)^2}{n_i} = X^2 \sim \chi^2_{K-1-d}$$