RESEARCH GOALS

FRANKLIN GAVILANEZ

University of Maryland, College Park, MD, USA

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My research is related to Network Tomography.

The scientifc value of this subject can be measured from the positive impact of tomography in society. In fact, tomography using CT scans and MRI scans is now well-known as a medical diagnostic tool which allows for detection of tumors and other abnormalities in a noninvasive way, providing very detailed images of the inside of the body using low dosage X-rays and magnetic fields. They have both also been used for determination of material defects in moderate size objects. There are many situations where one wants to monitor the electrical conductivity of different portions of an object, for instance, to find out whether a metal object, possibly large, has invisible cracks. This kind of tomography, usually called Electrical Impedance Tomography or EIT, has also medical applications like monitoring of blood flow[10], [11]. These diagnostic tools crucially depend on the mathematics of the Radon transform. A question that has arisen in the recent past is whether there is similar "tomographic" method to monitor the "health" of networks like the internet network, communications networks, highways, ATM, phone networks or resistors networks. In other words, can the problem of discovering the detailed inner structure of a network from the collection of end to end measurements be seen as a type of inverse problem?. The problem is to be able to find out whether a network, for instance a communications networks, is suffering some sort of breakdown. By that we mean that traffic along the network either can not reach every node in the network, or when we add a measure of traffic around the nodes, the traffic is so large in some parts of the network that it would take very long to go from one node to another. When the network is large, the information is naturally gathered at the "boundary" (outer nodes) of the network and hence the name *network tomography*. The similarity to conventional tomography becomes closer when one examines traffic "packets" sent from the boundary of the network to check whether they reach boundary . When the packets do arrive at other boundary points, we could also keep track of how long it took for the information packets to reach a given boundary point. Another way of monitoring a network, like the internet network, is done by using probes like sending an empty message to check whether a given destination exists allowing us to gather information about the network state. The answer to the above question is already known to be true on particular types of networks. For instance, if we consider the case of a square resistors network (lattice), EIT ideas can in fact effectively be used in this context [4]. Since a network can be modelled as a graph, some authors are trying to extend the results that exist on particular networks to arbitrary planar finite weighted graphs [3].

At this point, it is necessary to mention some work of C.A. Berenstein in [1]. Here is the general idea on how tomography can be used to do such monitoring of networks and implemented as a diagnostic tool in medicine and material sciences, some examples of tomography are given and also how the Radon transform is related to tomography. Consider open subsets A and Ω in $\mathbb{R}^2$ such that $A \subset\subset \Omega$ and let $f \in C_o(\mathbb{R}^2)$ be such that $Supp f \subset A$. Assume A represents an object from which we want to get some information.

Let Φ be the collection of all the straight lines in Ω connecting any pair of points a and b where $a, b \in \partial\Omega$, $a \neq b$, i.e., $\Phi = \{\zeta \; line \; / a, b \in \zeta \cap \partial\Omega, \; a \neq b\}$, and $f(x)$ represents the distribution or density at the point $x \in \Omega$. Let the Radon transform $R(f) = \left\{ \int\limits_{\zeta \in \Phi} f(x)dx \right\}$ be the set of all line integrals of the function f. The concept of *tomography* then can be understood as the reconstruction of the function f from the set of values given by $R(f)$. To recover f is thus the same as finding the inverse of the operator R . Hence, we are able to know the value of f at any point x in Ω. An standard inversion formula for the Radon transform in $\mathbb{R}^2$ is given by $f = F_2^{-1} F_1(Rf)$ where F_2 and F_1 stand for the 2-dimensional and 1-dimensional Fourier transform respectively. Another inversion formula that exists in terms of the adjoint R^* of the operator R is given by

$$\int_{\mathbb{R}^2} e^{i2\pi x \cdot \varsigma} \frac{|\varsigma|}{2} F_2(R^* Rf)(\varsigma) d\varsigma = f(x)$$

Using the fact that the 2-dimensional Fourier transform of the Laplacian operator is $\frac{|\varsigma|^2}{2}$, then one can introduce the operator Λ, square root of the Laplacian operator Δ, and we have

$$\Lambda(R^* Rf)(x) = f(x)$$

which is usually called *the backprojection inversion formula*.

An example of tomography is transmission CT (computerized tomography) in diagnostic radiology. Essentially the setup consists of a detector and an X-ray beam source. A cross-section of the human body is scanned by a thin X-rays beam. Given that the X-rays go through human tissue, it is clear the X-rays absorption is related to the attenuation coefficient. Let $f(x)$ be the X-ray attenuation coefficient of the tissue at the point x. Let ζ be the straight line representing the beam, I_o the initial intensity of the beam, and I_1 its intensity after having traversed the body. It follows from it that $\frac{I_1}{I_o} = \exp\{-\int\limits_{\zeta} f(x)dx\}$ thus the scanning process provides us with the line integral of the function f along each of the lines ζ. From the knowledge of all of these integrals (the Radon transform of f) the problem is to reconstruct f. Another example is MRI, magnetic resonance tomography. Here the situation is similar with the exception that the underlying space is $\mathbb{R}^3$ and the integrals take place over the family of all planes in $\mathbb{R}^3$. The example that is of more interest to me is the one of EIT. The question is to determine the electrical conductivity profile of a planar conducting plate from measurements on the boundary of the voltage induced by arbitrary currents also applied to the boundary. Let D the unit disk in $\mathbb{R}^2$ and β an strictly positive function defined on $\overline{D}$ which is unknown and represents the conductivity distribution inside the disk. Let Ψ be a given integrable function representing currents introduced at the boundary ∂D and such that the average of the values of Ψ on ∂D is zero, $\int_{\partial D} \Psi ds = 0$, and consider the boundary problem with Neumann conditions

$$\begin{cases} div(\beta grad\, u) = 0, & in\ \ D \\ \beta \frac{\partial u}{\partial n} = \Psi, & on\ \ \partial D \end{cases}$$

where n is the outer unit normal vector on ∂D. This problem has a unique solution u up to an additive constant. So, for Ψ given and β there exists a solution u. This defines a mapping $\beta, \Psi \mapsto \frac{\partial u}{\partial s}$ where s represents the tangent vector to ∂D and β is the only remaining function to be found. Let $\Lambda_\beta : \Psi \mapsto \frac{\partial u}{\partial s}$ be the linear operator that is determined by β. Given that β is to be found, then we consider the nonlinear mapping $\beta \mapsto \Lambda_\beta$ which is invertible. Now the problem (Calderón's problem) consists in determining β once Λ_β is given, i.e., the inverse of the above mapping, and this problem is called the *inverse conductivity problem*. An approximate solution is based on the Radon transform and it can be found in [11]. The situation here is that the physics of the disk implies that the lines of the currents flowing through the disk are geodesics and we measure the average of the conductivity along this geodesic, which is the Radon transform. Another kind of tomography is Internet tomography and [7] gives an introduction to this field. Here is shown how link-level parameters estimation can be obtained from path measurements.

Below is explained how EIT ideas can effectively be used in the context of particular networks. See the work of E. B. Curtis, and J. A. Morrow [2].

There is another analogy to mathematical tomography that arose in the context of electrical networks. Curtis and Morrow have done very interesting work in this context, both theoretical and in simulations [2] and [4]. This paper explains the concept of the Dirichlet to Neumann map and the EIT problem in the context of particular networks. When considering the very particular case of a square network of resistors, [4] provides with some relations and properties that characterize the Dirichlet to Neumann map. In [2] is shown that the conductance in the links of a rectangular resistors network can be determined by measurements at the boundary of the voltages generated by imposed currents (Neumann data). An *algorithm* for using the boundary measurements to compute the conductances is also given. These particular networks can be modelled by means of weighted graphs, where a weight ω plays here the role of the conductance (or conductivity). In general, given a network Ω (graph) with a pattern of traffic measured as the "usual" load between adjacent nodes, a Laplace operator Δ_ω is associated. A weight ω is a sequence of values representing the usual loads between every pair of adjacent nodes in the network which is thought as a function on the links of the network. Examples are connectivity, delay, packet loss, information traffic. In addition, a discrete Laplace operator on the graph is associated to each weight ω, via the consideration of ω-harmonic functions. Indirectly, via an associated potential function u (a function on the nodes of the graph) the so-called Neumann to Dirichlet map Λ_ω for each Laplacian Δ_ω is defined and the solution of Laplace's equation on the graph subject to Dirichlet and Neumann boundary conditions is considered. The discrete version of the Laplace operator is defined as follows. Let Ω_o be the set of nodes, Ω_1 the set of edges or links and $\omega : \Omega_1 \to \mathbb{R}^+$ the weight function . For any function $u : \Omega_o \to \mathbb{R}$, the function $\Delta_\omega u : int\, \Omega_o \to \mathbb{R}$, where $int\, \Omega_o$ is the interior of Ω_o, is defined by $\Delta_\omega u(p) = \sum_{q \in N(p)} \omega(pq)(u(q) - u(p))$ with pq the edge having p and q as endpoints, $\omega(pq)$ the weight in the edge pq and $N(p)$ the neighborhood of p which is nothing but the set of nodes p connected to q by an edge.The main result here is the solution of the inverse conductivity problem for a particular network of resistors. The term tomography appears here by analogy. The approach here gives a direct method for calculating the conductivity ω of each resistor in the network. The following fundamental questions for resistors networks are investigated

(i) Is the map Λ_ω to ω one-to-one?

(ii) Is there a constructive algorithm to obtain ω from Λ_ω?

(iii) What type of boundary measurements and associated probes we need to construct Λ_ω from the available boundary data and/or probes?

Once these results are studied, a question that arises is if is it possible to extend these results to general finite planar weighted graphs?. For instance, is (i) true?. In [3] we can see that the answer is yes. In fact, the next subsection shows with a theorem that the Neumann-to-Dirichlet map for $\Delta_{\omega'}$ is different to that for Δ_ω. On the other hand, while (i) and (ii) can be considered as corollaries of this theorem and its proof, there is the open question of how to do it effectively for non square lattices networks en we refer to the work of S-Y. Chung and C. A. Berenstein [3].

The analytical method that is being developed for general finite weighted planar graph tomography problems is based on [1], [2], and [4]. It will lead to approximate solutions involving tomography on trees. As it turns out, inversion formula for the Radon transform on trees is already known and it can be found in [5]. A line in a tree is the shortest path between two points; rules out closed circuits. Formulation of solutions to some of the network tomography problems can be described using tools similar to the Radon transform for trees. In trees, the objective is to reconstruct a function f on the tree (actually f is defined on the nodes of the tree). For each path l (line), $Rf(l)$ is the sum of the values of f along the nodes of the line. It is possible to reconstruct f from $Rf(l)$ by using inversion formulae for the Radon transform on trees: they involve the discrete analog of the Laplace operator and the Neumann to Dirichlet map Λ_ω for a given weight ω on the edges of the graph similar to the case in [2], and [4] for particular networks. For the sake of completeness I give the definition of the discrete Laplace operator and other definitions that are used in [3]. In this case the network is modelled in the following way. We have a collection of nodes and edges between the nodes in a finite planar connected graph G. We denote by V the set of nodes of G and by E the set of edges of G. Usually, the graph G is denoted by $G(E, V)$. A particular subset of this graph G is denoted by ∂G and called the boundary of G. In the context of [3] these are the nodes accessible to whoever is trying to monitor the traffic in G. The boundary edges are those links whose two endpoints are in ∂G. It is assumed that G remains connected even if the boundary edges are removed. Here the boundary edges play no role, thus the authors may as well assume that there are none. It is also assumed that ∂G is not empty.

Furthermore, it is assumed that to every edge in E there is an associated nonnegative number $\omega(x, y)$ which corresponds to the *traffic* between the endpoints x and y of the edge. The degree $d_\omega x$ of a node x in the weighted graph G with weight ω is defined for $x \in V$ by $d_\omega x = \sum_{y \in V} \omega(x, y)$ and the Laplacian operator corresponding to this weight ω is defined by $\Delta_\omega f(x) = \sum_{y \in V}[f(y) - f(x)] \cdot \frac{\omega(x,y)}{d_\omega x}$. A graph

$S = S(V', E')$ is said to be a *subgraph* of $G(E, V)$ if $V' \subset V$ and $E' \subset E$. In this case, we call G a *host graph* of S. The integration of a function $f : G \to \mathbb{R}$ on a graph $G = G(V, E)$ is defined by $\int_G f = \sum_{x \in V} f(x) d_\omega x$ or simply $\int_G f d_\omega$. For a subgraph S of a graph $G = G(V, E)$ the (node) *boundary* ∂S of S is defined to be the set of all nodes $z \in V$ not in S but adjacent to some node in S, i.e., $\partial S = \{z \in V \mid z \notin S$ and $z \sim y$ for some $y \in S\}$ and the *inner boundary* $\overset{\circ}{\partial} S$ by $\overset{\circ}{\partial} S = \{z \in S \mid y \sim z$ for some $y \in \partial S\}$ where $z \sim y$ means that the two nodes z and y are connected by an edge in E. Also, by $\overline{S}$ we denote a graph whose nodes and edges are in $S \cup \partial S$. The (outward) normal derivative $\frac{\partial f}{\partial n_\omega}(z)$ at $z \in \partial S$ is defined to be $\frac{\partial f}{\partial n_\omega}(z) = \sum_{y \in S}[f(z) - f(y)] \cdot \frac{\omega(z,y)}{d'_\omega z}$, where $d'_\omega z = \sum_{y \in S} \omega(z, y)$.

In this model, there are two kinds of disruptions of traffic data that could arise. In one of them, disruptions occurs when an edge "ceases" to exist, in this case the "topology" of the graph changes, and I refer to the important work of Fan Chung and her collaborators [6] which offers crucial insights into this question. In the other, the weights change because of "increase" of traffic and the network configuration remains the same. In this second situation, we can appeal to the following theorem that also shows that the map Λ_ω is one-to-one for general finite planar weighted connected graphs (the same as (i) in [2]).

Theorem 1 *Let ω_1 and ω_2 be weights with $\omega_1 \leq \omega_2$ on $\overline{S} \times \overline{S}$, and $f_1, f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying for $j = 1, 2$,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0, \ x \in S \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \Phi(z), \ z \in \partial S \\ \int_S f_j d_{\omega_j} = K \end{cases}$$

for any given function $\Phi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \Phi = 0$, and a given constant K with $K > m_0$, where $m_0 = \max_{j=1,2} |m_j| \cdot vol(S, w_j)$, $m_j = \min_{z \in \partial S} f_j(z)$, $j = 1, 2$ and $vol(S, w_j) = \sum_{x \in S} d_{\omega_j} x$. If we assume that

$$(i)\, \omega_1(z, y) = \omega_2(z, y) \text{ on } \partial S \times \overset{\circ}{\partial} S$$
$$(ii)\, f_1|_{\partial S} = f_2|_{\partial S},$$

then we have

$$f_1 \equiv f_2$$

and

$$\omega_1(x, y) = \omega_2(x, y)$$

for all x and y in $\overline{S}$.

The condition that $\Delta_\omega f(x) = 0$ corresponds to the fact that the value $f(x)$ is the weighted average of the values of f at the adjacent nodes

We conclude that the data distinguishes the two cases. That is, it is possible to decide whether there is an increase of traffic (ω) somewhere in the network or not. While this is only a uniqueness theorem, nevertheless, we can effectively compute the actual weight from the knowledge of the Dirichlet data for convenient choices of the input Neumann data in a way similar to that done in [2] and [4] for lattices. Similarly, the Green function of this Neumann boundary value problem can be represented by an explicit matrix. In order to deal with this inverse problem, it is necessary at least to know or be given the boundary data such as $f(z)$, $\frac{\partial f}{\partial n_\omega}(z)$ for $z \in \partial S$ and ω near the boundary. So it is natural to assume that $f\,|_{\partial S}$, $\frac{\partial f}{\partial n_\omega}|_{\partial S}$ and $\omega|_{\partial S \times \overset{\circ}{\partial S}}$ are known (given or measured). The general case is under investigation and constitutes my expected research area along with the developing of the mathematical tools to represent the Green function. In addition, no work on numerical solution to the inverse problem has been done yet. Here, [6] is used as a primary consultation book, specially the section devoted to weighted Laplacians. Material from the seminar-courses [**?**] is useful in understanding the theory of weighted graphs. Within my expected research area important networks to be considered are communications networks and therefore analysis of data traffic needs to be done for the monitoring of the networks. For instance, the weights that are mentioned above can correspond to an increase of traffic (of messages) in a communication network, congestion or attacks.

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