RHEA
A Relativistic Hydrodynamics code for Exploring Active Galactic Nuclei

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April 30, 2003

Introduction

- Galaxies exist in clusters often dominated by a central radio galaxy with a massive black hole at its core referred to as the *Active Galactic Nucleus* (AGN)

- The AGN emits highly relativistic jets of matter into the surrounding *Intra-Cluster Medium* (ICM).

- These have become objects of recent study due to the X-ray telescopes *Chandra* and *XMM-Newton*.

Mathematical Model

- Consider homogeneous fluid in region of space gravitationally bound by ambient dark matter.

- Solve equations of special relativistic hydrodynamics (SRHD) in this region.

- Inject jets by use of appropriate boundary conditions.

Motivation

- It will contribute to our knowledge of black holes and their impact on the surround environment.
- The impact of the jets will have an affect on the properties of the cluster which in turn affect cosmological arguments.
- It will assist in explaining discrepancies between model predictions and observational data.
- We will be able to discern certain information about the cluster, e.g. its age, by comparing observations with model simulations.

Equations of SRHD

Index notation: $\alpha, \beta = 0, 1, 2, 3$ and $i, j = 1, 2, 3$.
Naturalized units i.e. $c = 1$.

$$\frac{\partial}{\partial x^\alpha}(\rho u^\alpha) = 0$$
$$\frac{\partial}{\partial x^\beta}(T^{\alpha\beta}) = 0$$

$T^{\alpha\beta}$ - stress-energy tensor.
u^α - 4-velocity.
ρ - proper rest mass energy.

$$u^0 = \Gamma$$
$$u^i = \Gamma v^i$$

v^i - 3-velocity.
$\Gamma = (1 - v^i v_i)^{-1/2}$ - Lorentz factor.

For a perfect fluid

$$T^{\alpha\beta} = \rho h u^{\alpha} u^{\beta} + p g^{\alpha\beta}$$

$$h = 1 + e + \frac{p}{\rho}$$

p - fluid pressure.
$g^{\alpha\beta}$ - metric tensor.
h - enthalpy.
e - specific internal energy.

Equations of SRHD in Conservation Form

For purposes of computation equations can be rearranged into vector conservative flux form.

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^i(\mathbf{U})}{\partial x^i} = 0$$

$$\mathbf{U} = \begin{bmatrix} D \\ S^1 \\ S^2 \\ S^3 \\ \tau \end{bmatrix} \quad \mathbf{F}^i = \begin{bmatrix} Dv^i \\ S^1 v^i + p\delta^{1i} \\ S^2 v^i + p\delta^{2i} \\ S^3 v^i + p\delta^{3i} \\ S^i - Dv^i \end{bmatrix}$$

D - rest mass density.
S^i - momentum density.
τ - energy density.

Conserved and Primitive Variables

Conserved variables, D, S^i, τ are related to primitive variables ρ, v^i, e via the non-linear relations

$$\begin{aligned} D &= \rho \Gamma \\ S^i &= \rho h \Gamma^2 v^i \\ \tau &= \rho h \Gamma^2 - p - D \end{aligned}$$

We also need the equation of the state for the gas. The equation of state for an ideal polytropic gas is used:

$$p = (\gamma - 1)\rho e$$

γ - adiabatic constant.

These will need to be solved at least once at each time step.

Non-Relativistic Limit

The non-relativistic limit is described by $|\mathbf{v}| \approx 0$ and $h \approx 1$. Under such conditions the equations of SRHD are well approximated by Euler's equaitons:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^i(\mathbf{U})}{\partial x^i} = 0$$

$$\mathbf{U} = \begin{bmatrix} \rho \\ v^1 \\ v^2 \\ v^3 \\ e \end{bmatrix} \quad \mathbf{F}^i = \begin{bmatrix} \rho v^i \\ \rho v^2 + p\delta^{1i} \\ \rho v^2 + p\delta^{2i} \\ \rho v^2 + p\delta^{3i} \\ v^i(\frac{1}{2}\rho v^2 + \rho e) \end{bmatrix}$$

The significance of this is that RHEA can make use of non-relativistic results as a means of verification.

Coordinate System

A Cartesian coordinate system is used.

Advantages:

- Underlying equations are more simply expressed and numerical methods more simply implemented in Cartesian coordinates.
- Test cases, e.g. shock tube problem, are naturally suited to a Cartesian grid.

Disadvantages:

- In area of application many quantities vary radially making spherical coordinates a better choice.
- Overall problem is axi-symmetric about the jets which can not be taken advantage of in Cartesian coordinates.

Method Of Lines

Spacially discretize the conservation law to leave a system of ODEs.

$$\begin{aligned}\frac{d\mathbf{U}_{ijk}}{dt} = & -\frac{1}{dx}(\mathbf{F}_{i+1/2,j,k} - \mathbf{F}_{i-1/2,j,k}) \\ & -\frac{1}{dy}(\mathbf{F}_{i,j+1/2,k} - \mathbf{F}_{i,j-1/2,k}) \\ & -\frac{1}{dz}(\mathbf{F}_{i,j,k+1/2} - \mathbf{F}_{i,j,k-1/2})\end{aligned}$$

$\mathbf{U}_{ijk}$ - Cell centered averages of conserved quantities.
$\mathbf{F}_{i\pm1/2}, \mathbf{F}_{i,j\pm1/2,k}, \mathbf{F}_{i,j,k\pm1/2}$ - Averaged flux values at cell interfaces.

Move forward in time using an ODE solver. RHEA uses RK2 and RK4.

To calculate flux at cell boundaries solve Riemann problem. RHEA uses *Lax-Friedrichs* and HLL.

Lax-Frierichs Method

Flux across a cell boundary is given by

$$\mathbf{F}_{BDY} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - C\frac{h}{2k}(\mathbf{U}_R - \mathbf{U}_L)$$

h - spacial discretization.
k - temporal discretization.

Second term is an artificial viscosity whose influence can be controlled by the parameter C.

Relativistic HLL Method

Flux across a cell boundary is given by

$$\mathbf{F}_{BDY} = \frac{a_R^+ \mathbf{F}_L - a_L^- \mathbf{F}_R + a^+ a^- (\mathbf{U}_R - \mathbf{U}_L)}{a_R^+ - a_L^-}$$

$$a_L^- = \mathrm{min}\{0, a_L\}, \quad a_R^+ = \mathrm{max}\{0, a_R\}$$

$$a_R = \frac{\bar{v} + \bar{c}_s}{1 + \bar{v}\bar{c}_s}, \quad a_L = \frac{\bar{v} - \bar{c}_s}{1 - \bar{v}c}$$

Barred quantities represent averages over the left and right cells.

Lax-Wendroff Scheme

Predictor-corrector method on staggered grid . Predictor is Lax-Freidrichs scheme. In 2-dimensions:

$$
\begin{aligned}
\mathbf{U}^{n+1/2}_{i+1/2,j+1/2} = & \frac{1}{4}(\mathbf{U}^n_{i,j} + \mathbf{U}^n_{i+1,j} + \mathbf{U}^n_{i,j+1} + \mathbf{U}^n_{i+1,j+1}) \\
& -\frac{dt}{2dx}(\mathbf{F}^{n+1/4}_{i+1,j+1/2} - \mathbf{F}^{n+1/4}_{i,j+1/2}) \\
& -\frac{dt}{2dx}(\mathbf{G}^{n+1/4}_{i+1/2,j+1} - \mathbf{G}^{n+1/4}_{i+1/2,j})
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{F}^{n+1/4}_{i,j+1/2} = & \mathbf{F}^1(\frac{1}{2}(\mathbf{U}^n_{i,j+1} + \mathbf{U}^n_{i,j}) \\
& - \frac{dt}{4dx}(\mathbf{F}^2(\mathbf{U}^n_{i,j+1} - \mathbf{F}^2(\mathbf{U}^n_{ij}))
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{G}^{n+1/4}_{i+1/2,j} = & \mathbf{F}^2(\frac{1}{2}(\mathbf{U}^n_{i+1,j} + \mathbf{U}^n_{i,j}) \\
& - \frac{dt}{4dy}(\mathbf{F}^1(\mathbf{U}^n_{i+1,j} - \mathbf{F}^1(\mathbf{U}^n_{ij}))
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^{n} - \\
&\frac{dt}{2dx}(\mathbf{F}^1(\mathbf{U}_{i+1/2,j+1/2}^{n+1/2}) + \mathbf{F}^1(\mathbf{U}_{i+1/2,j-1/2}^{n+1/2}) \\
&\quad - \mathbf{F}^1(\mathbf{U}_{i-1/2,j+1/2}^{n+1/2}) + \mathbf{F}^1(\mathbf{U}_{i-1/2,j-1/2}^{n+1/2}) - \\
&\frac{dt}{2dy}(\mathbf{F}^2(\mathbf{U}_{i+1/2,j+1/2}^{n+1/2}) + \mathbf{F}^2(\mathbf{U}_{i-1/2,j+1/2}^{n+1/2}) \\
&\quad - \mathbf{F}^2(\mathbf{U}_{i+1/2,j-1/2}^{n+1/2}) + \mathbf{F}^2(\mathbf{U}_{i-1/2,j-1/2}^{n+1/2})
\end{aligned}
$$

This is a second order scheme.

Boundary Conditions

Boundary conditions are enforced by one layer of ghost cells surrounding the region.

Inflow Ghost cells are set to appropriate values at the beginning of each time step.

Outflow Ghost cells are set to equal the value of neighboring cells at each time step.

Reflective Ghost cells are set to equal the value of neighboring cells except for relevant component of velocity which has its direction reversed.

CFL Condition

To ensure stability in time the CFL condition must be satisfied. This is done globally in RHEA by ensuring

$$dt < \min_n \left\{ \min_i \left\{ \frac{dx^i}{\max\{v^i, c_s\}} \right\} \right\}$$

i - component index.
n - cell index.
c_s - sound speed.

Recovery of Primitive Variables

At each time step we need the primitive variables in order to calculate the fluxes. This can be achieved by finding the roots of the equation of state.

$$f(p) = (\gamma - 1)\rho^* e^* - p$$

$$\rho^* = D/\Gamma$$

$$e^* = \frac{\tau + D(1 - \Gamma) + p(1 - \Gamma^2)}{D\Gamma}$$

$$\mathbf{v}^* = \frac{\mathbf{S}^2}{\tau - D - p}$$

$$\Gamma = \frac{1}{\sqrt{1 - \mathbf{v}^{*2}}}$$

RHEA has two methods of finding these roots - *Newton-Raphson* method and the method of *Bisection*. In the Newton-Raphson method the derivative of $f(p)$ is approximated by

$$f'(p) = |\mathbf{v}^*|^2 c_s^2 - 1$$

$$c_s = \sqrt{\frac{(\gamma - 1)\gamma e}{1 - \gamma e}}$$

The pressure on the previous time step can be used as an initial guess.
We also impose the constraint

$$p > \mathbf{S} - \tau - D$$

to ensure that $|\mathbf{v}| < 1$.

The monotonic behavior of f is used to find bounds for the bisection method.

Shocktube Problem - 1D

Sod - 1978

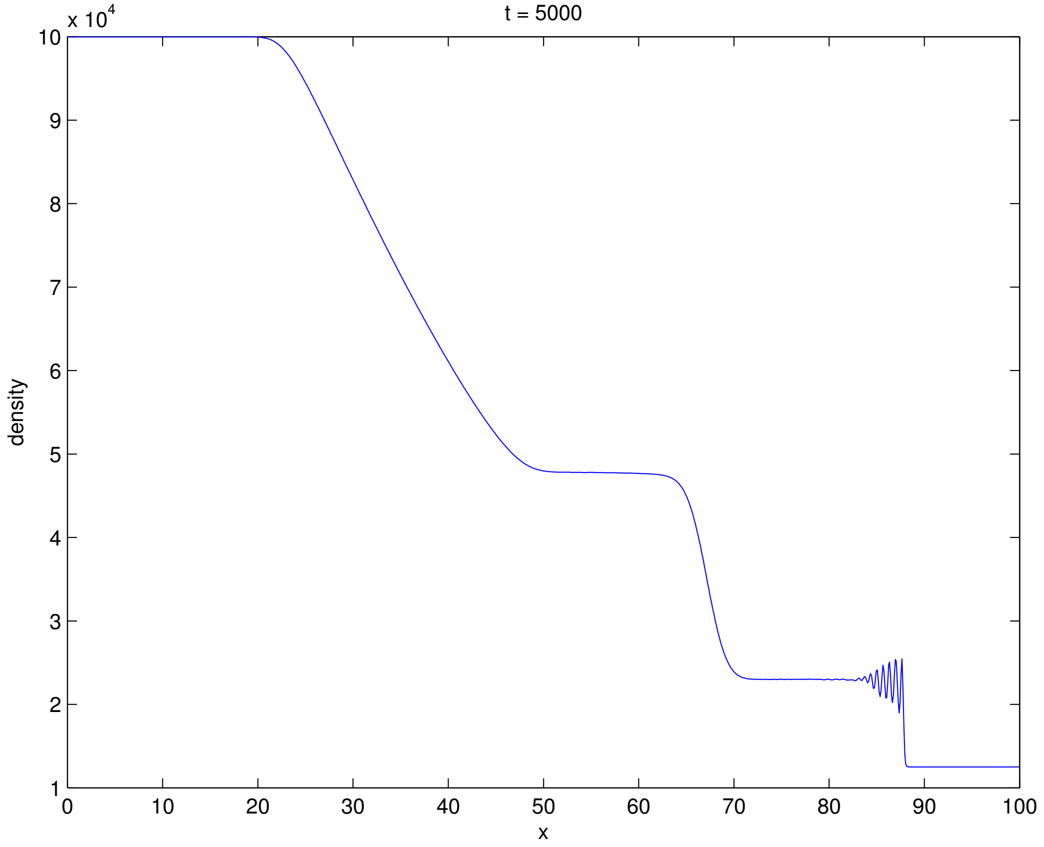
$$\begin{array}{ll} \rho_l = 1 & \rho_r = 0.125 \\ v_l = 0 & v_r = 0 \\ e_l = 2.5 & e_r = 2 \end{array}$$

$$\gamma = 1.4$$

Hawley, Smarr, Wilson - 1984

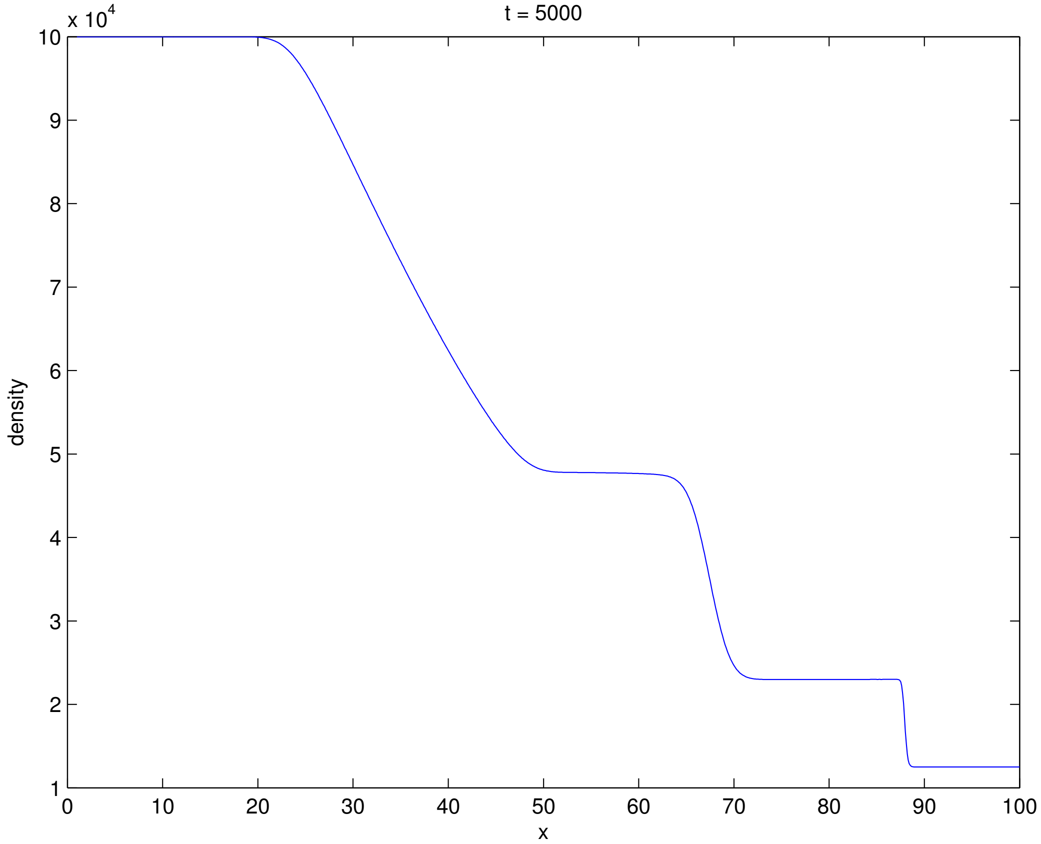
$$\begin{array}{ll} \rho_l = 1 & \rho_r = 0.125 \\ v_l = 0 & v_r = 0 \\ e_l = 2.5 \times 10^{-5} & e_r = 2 \times 10^{-5} \end{array}$$

$$\gamma = 1.4$$

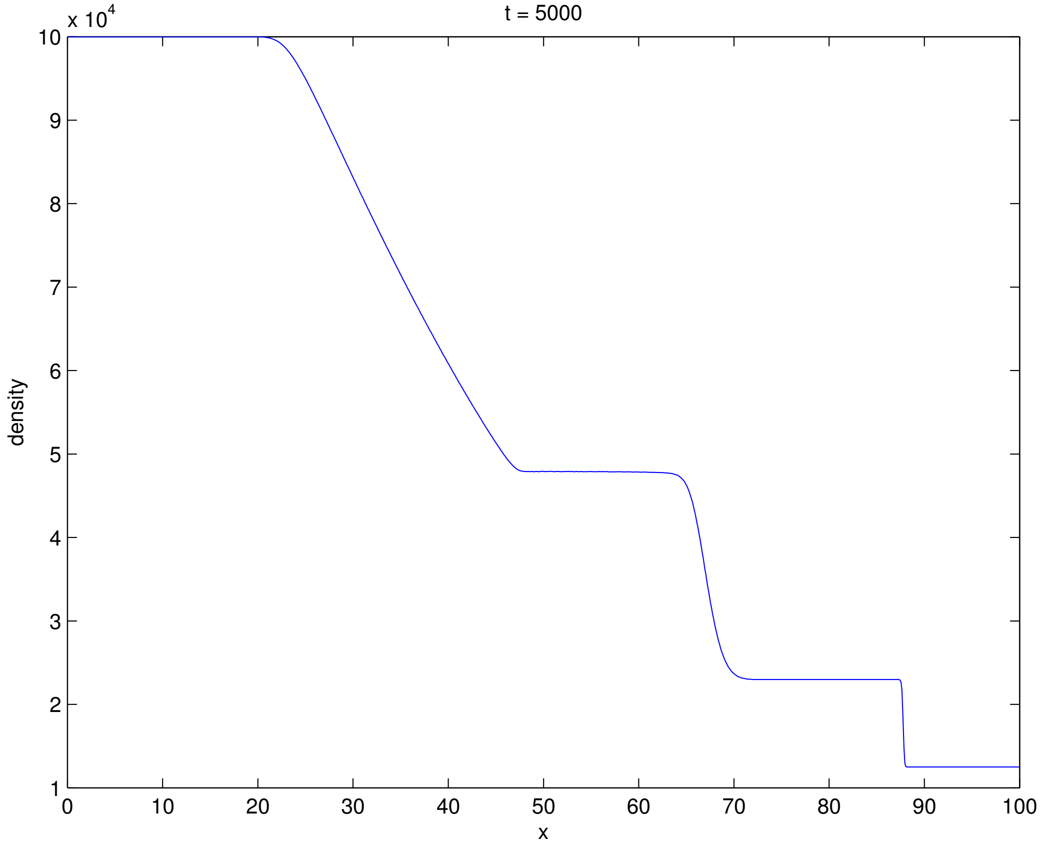


1-D non-relativistic shock tube

- Runge-Kutta 2
- Lax-Friedrichs method
- 1000 cells

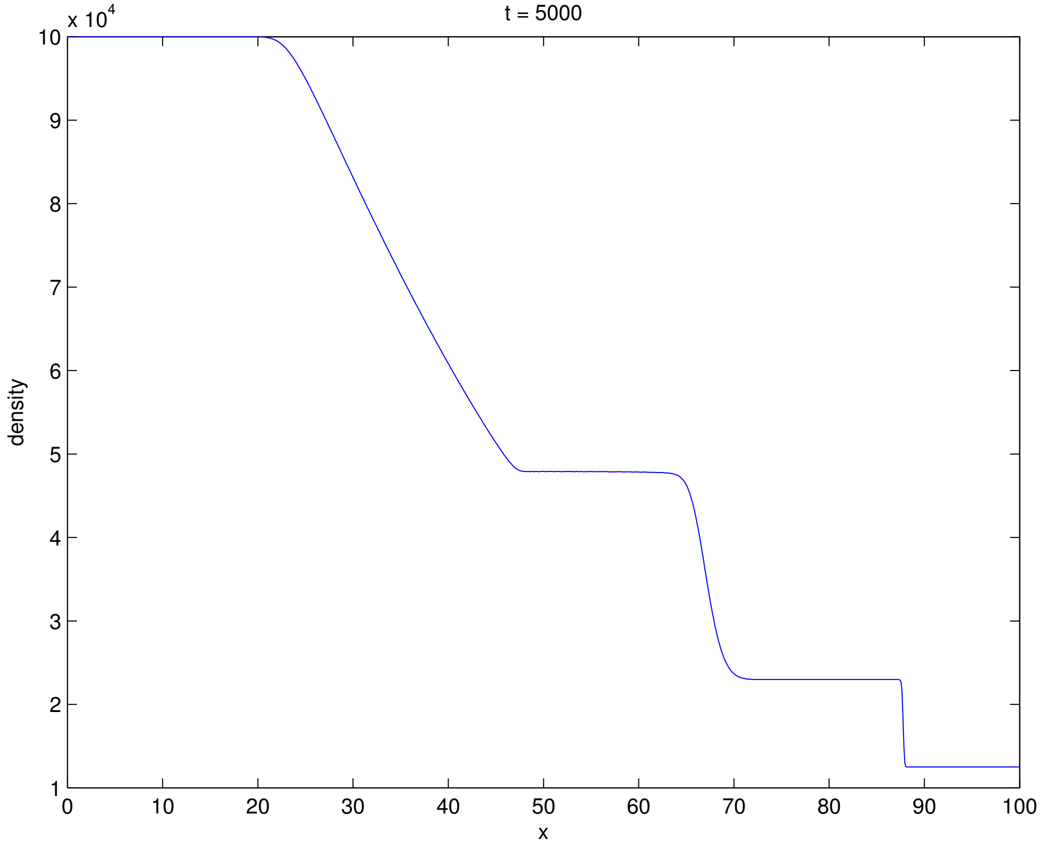


1-D non-relativistic shock tube
- Runge-Kutta 4
- Lax-Friedrichs method
- 1000 cellls



1-D non-relativistic shock tube

- Runge-Kutta 2
- HLL method
- 1000 cells



1-D non-relativistic shock tube

- Runge-Kutta 4
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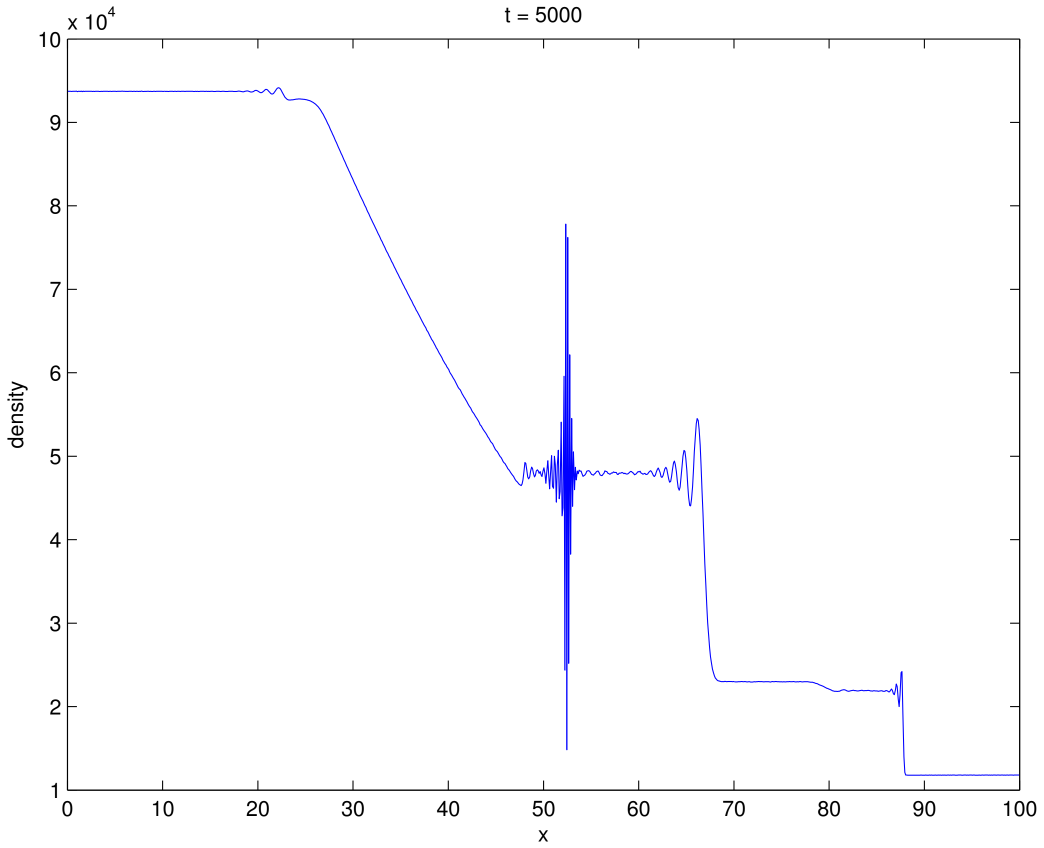


1-D non-relativistic shock tube

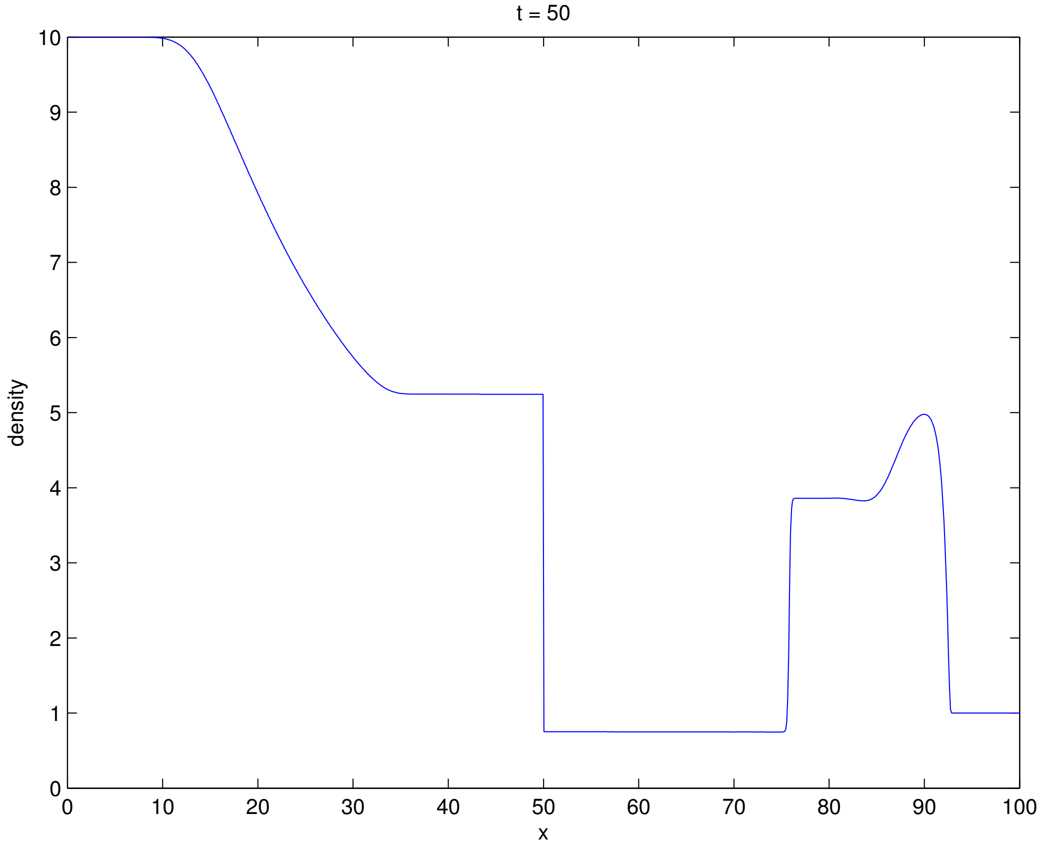
- Lax-Wendroff scheme
- 1000 cells

1-D Relativistic Shocktube Problem

Hawley, Smarr, Wilson - 1984

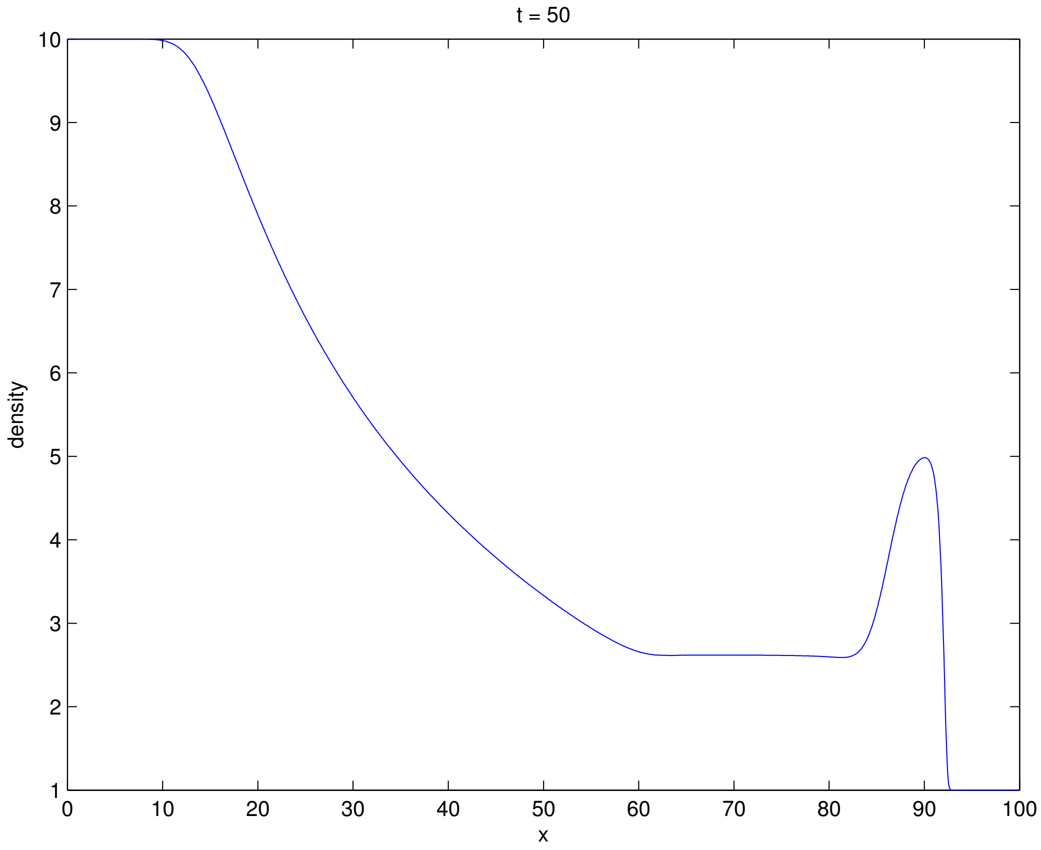
$$\begin{array}{ll} \rho_l = 10 & \rho_r = 1 \\ v_l = 0 & v_r = 0 \\ e_l = 2 & e_r = 10^{-6} \end{array}$$

$$\gamma = 5/3$$



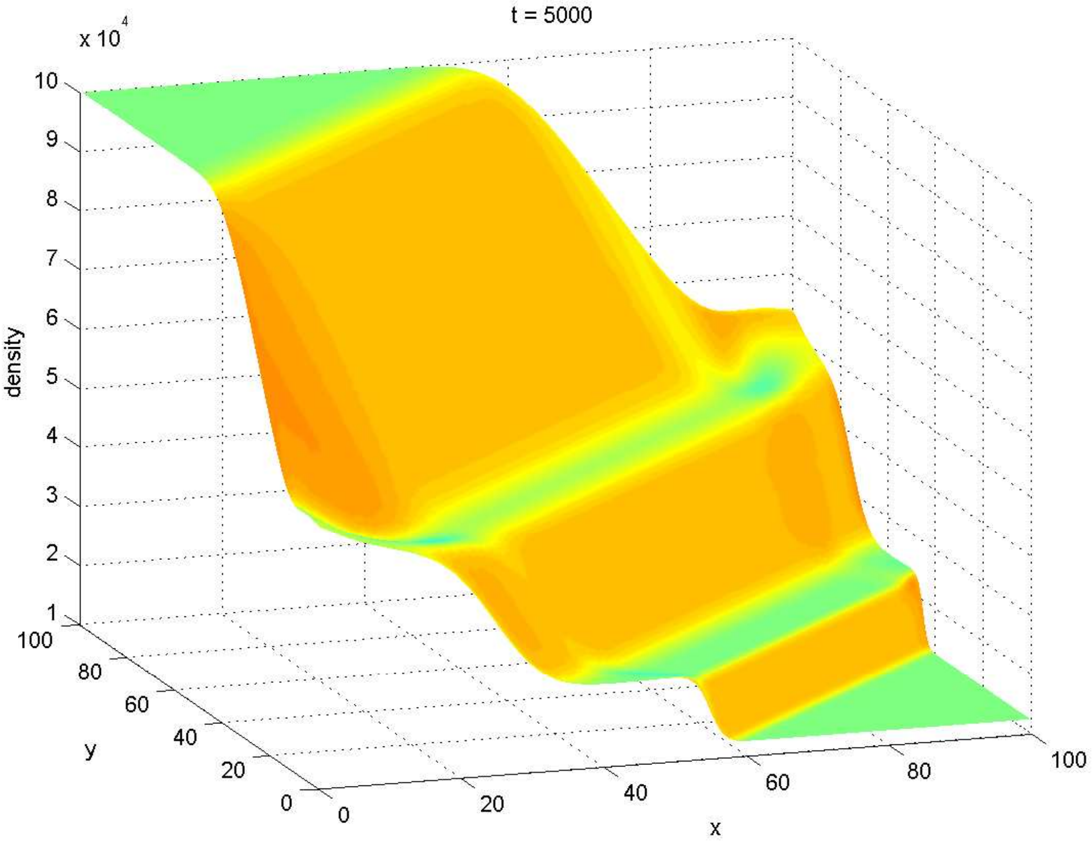
1-D relativistic shocktube

- Runge-Kutta 4
- HLL method
- 1000 cells



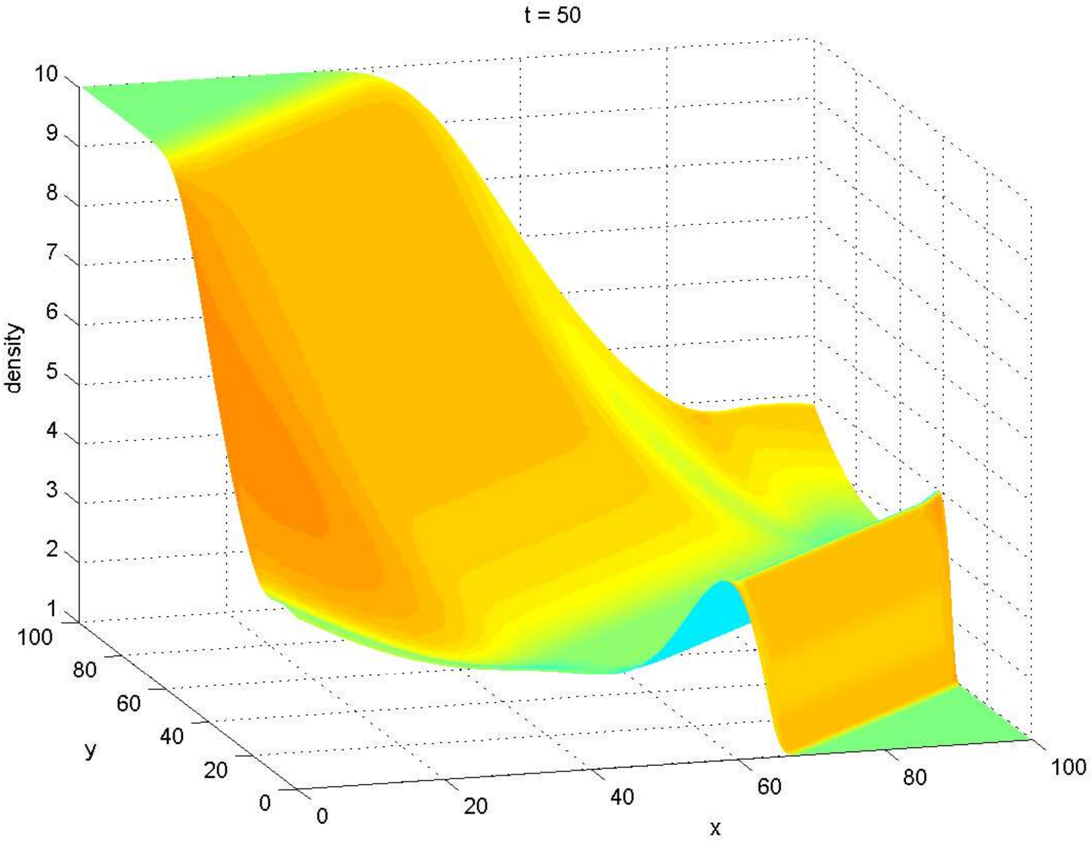
1-D relativistic shocktube

- Runge-Kutta 4
- Lax-Friedrichs method
- 1000 cells



2-D non-relativistic shocktube

- Runge-Kutta 4
- Lax-Friedrichs method
- 200 × 200 cells



2-D relativistic shocktube

- Runge-Kutta 4
- Lax-Friedrichs method
- 200 × 200 cells

Where now?

First

- Resolve issure with shocktube problem.

Then

- Application
- 3-dimensions
- Parallelize