A GIAMBELLI FORMULA FOR ISOTROPIC GRASSMANNIANS

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ABSTRACT. Let X be a symplectic or odd orthogonal Grassmannian parametrizing isotropic subspaces in a vector space equipped with a nondegenerate (skew) symmetric form. We prove a Giambelli formula which expresses an arbitrary Schubert class in $\mathrm{H}^*(X,\mathbb{Z})$ as a polynomial in certain special Schubert classes. We introduce and study *theta polynomials*, a family of polynomials which are positive linear combinations of products of Schur Q- and S-functions, and whose algebra agrees with the Schubert calculus on X.

0. Introduction

Let $\mathrm{G} = \mathrm{G}(m,N)$ denote the Grassmannian of m-dimensional subspaces of complex affine N-space. To each integer partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ whose Young diagram is contained in an $m \times (N-m)$ rectangle, we associate a Schubert class σ_λ in the cohomology ring of G. The *special* Schubert classes $\sigma_1, \ldots, \sigma_{N-m}$ are the Chern classes of the universal quotient bundle $\mathcal{Q}$ over $\mathrm{G}(m,N)$; they generate the ring $\mathrm{H}^*(\mathrm{G},\mathbb{Z})$. The classical *Giambelli formula* [G]

$$\sigma_\lambda = \det(\sigma_{\lambda_i+j-i})_{i,j} \tag{1}$$

is an explicit expression for σ_λ as a polynomial in the special classes; as is customary, we agree here and in later formulas that $\sigma_0 = 1$ and $\sigma_r = 0$ for $r < 0$.

The relation between the Schubert calculus on the Grassmannian $\mathrm{G}(m,N)$ and the algebra of Schur's S-functions s_λ (originally defined by Cauchy [C] and Jacobi [J]) is well known. Given $x = (x_1, x_2, \ldots)$ a countably infinite set of commuting independent variables, we define the elementary symmetric functions $e_r(x)$ by the formal relation

$$\prod_{i=1}^{\infty}(1+x_i t) = \sum_{r=0}^{\infty} e_r(x) t^r$$

and set, for any partition λ, $s_{\lambda'}(x) = \det(e_{\lambda_i+j-i}(x))_{i,j}$. Here λ' is the partition whose Young diagram is the transpose of the diagram of λ. The ring $\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$ of symmetric functions in x has a free $\mathbb{Z}$-basis consisting of the Schur functions s_λ, for all partitions λ. These Schur S-functions enjoy many good combinatorial properties, such as nonnegativity of their coefficients, and multiply exactly like the Schubert classes on $\mathrm{G}(m,N)$, when m and N are sufficiently large.

There is a closely analogous story to the above for the Lagrangian Grassmannian $\mathrm{LG}(n,2n)$ which parametrizes maximal isotropic subspaces of $\mathbb{C}^{2n}$, with respect to a symplectic form. The Schubert classes in $\mathrm{H}^*(\mathrm{LG},\mathbb{Z})$ are indexed by *strict* partitions, i.e., partitions with distinct (non-zero) parts, whose diagrams fit in a square of side

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n. The special Schubert classes $\sigma_r = c_r(\mathcal{Q})$ again generate the cohomology ring, and there is a Giambelli-type formula due to Pragacz [Pra]. This latter may be described in two steps: For partitions $\lambda = (a,b)$ with only two parts, we have

$$\sigma_{a,b} = \sigma_a\sigma_b - 2\sigma_{a+1}\sigma_{b-1} + 2\sigma_{a+2}\sigma_{b-2} - \cdots \tag{2}$$

while for λ with 3 or more parts,

$$\sigma_\lambda = \text{Pfaffian}(\sigma_{\lambda_i,\lambda_j})_{i<j}. \tag{3}$$

The identities (2) and (3) in fact also go back to the work of Schur [S], who considered a family of symmetric functions $\{Q_\lambda\}$ known as Schur Q-functions. We define $q_r(x)$ by the equation

$$\prod_{i=1}^{\infty} \frac{1+x_it}{1-x_it} = \sum_{r=0}^{\infty} q_r(x)t^r$$

and then use the same relations (2) and (3) with $q_r(x)$ in place of σ_r to define $Q_{a,b}(x)$ and then $Q_\lambda(x)$, for each strict partition λ. If we let $\Gamma = \mathbb{Z}[q_1, q_2, \ldots]$ denote the ring of Schur Q-functions, then the $\{Q_\lambda\}$ for λ strict form a $\mathbb{Z}$-basis for Γ, whose algebra agrees with Schubert calculus on $\text{LG}(n,2n)$, as $n \to \infty$. Moreover, there is a well developed combinatorial theory for the Q-functions, analogous to that for the S-functions.

Choose $k \geq 0$ and consider now the Grassmannian $\text{IG}(n-k,2n)$ of isotropic $(n-k)$-dimensional subspaces of $\mathbb{C}^{2n}$, equipped with a symplectic form. We call a partition λ *k-strict* if no part greater than k is repeated, i.e., $\lambda_j > k \Rightarrow \lambda_j > \lambda_{j+1}$. The Schubert classes on IG are indexed by k-strict partitions whose diagrams fit in an $(n-k) \times (n+k)$ rectangle. Given such a λ and a complete flag of subspaces $F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = \mathbb{C}^{2n}$ such that $F_{n+i} = F_{n-i}^{\perp}$ for $0 \leq i \leq n$, we have a Schubert variety

$$X_\lambda(F_\bullet) := \{\Sigma \in \text{IG} \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \quad \forall\, 1 \leq j \leq \ell(\lambda)\},$$

where $\ell(\lambda)$ denotes the number of (non-zero) parts of λ and

$$p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j : \lambda_i + \lambda_j > 2k + j - i\}. \tag{4}$$

This variety has codimension $|\lambda| = \sum \lambda_i$ and defines, using Poincaré duality, a Schubert class $\sigma_\lambda = [X_\lambda(F_\bullet)]$ in $\text{H}^{2|\lambda|}(\text{IG},\mathbb{Z})$. As above, we consider the special Schubert classes $\sigma_r = [X_r(F_\bullet)] = c_r(\mathcal{Q})$ for $1 \leq r \leq n+k$.

In [BKT1], we proved a Pieri rule for the products $\sigma_r\sigma_\lambda$ in $\text{H}^*(\text{IG})$. Equipped with this rule and the help of a computer, we observed that (i) when $\lambda_j \leq k$ for all j, then σ_λ is given by the determinantal formula (1); (ii) when $\lambda_j > k$ for all non-zero λ_j, then λ is strict and σ_λ is given by the Pfaffian formulas (2), (3). It is tempting to ask for an analogous Giambelli formula for σ_λ when λ is a general k-strict partition. Note that the formula is determined only up to an ideal of relations; whatever the answer, it must naturally interpolate between the Jacobi-Trudi determinant (1) and the Schur Pfaffian (3). This question was also raised by Pragacz and Ratajski [PR], who were using a different set of special Schubert classes.

The answer we give depends crucially on our choice of k-strict partitions to index the Schubert classes, and uses Young's *raising operators* [Y, p. 199]. For any integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finite support and $i < j$, we define $R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots)$; a raising operator R is any monomial in these R_{ij}'s. Set $m_\alpha = \prod_i \sigma_{\alpha_i}$ and $R\, m_\alpha = m_{R\alpha}$ for any raising operator R [1]. Using these operators, the Giambelli formulas (1) and (2)–(3) can be expressed as

$$\sigma_\lambda = \prod_{i<j}(1-R_{ij})\, m_\lambda \quad \text{and} \quad \sigma_\lambda = \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}}\, m_\lambda, \tag{5}$$

respectively (compare with the identities (6) below).

Definition 1. For a general k-strict partition λ, we define the operator

$$R^\lambda = \prod(1-R_{ij}) \prod_{\lambda_i+\lambda_j>2k+j-i}(1+R_{ij})^{-1}$$

where the first product is over all pairs $i<j$ and second product is over pairs $i<j$ such that $\lambda_i+\lambda_j>2k+j-i$.

Theorem 1. *For any k-strict partition λ, we have $\sigma_\lambda = R^\lambda m_\lambda$ in the cohomology ring of* $\mathrm{IG}(n-k,2n)$.

For example, in the ring $\mathrm{H}^*(\mathrm{IG}(4,10))$ (where $k=1$) we have

$$\sigma_{321} = \frac{1-R_{12}}{1+R_{12}}(1-R_{13})(1-R_{23})\, m_{321} = (1-2R_{12}+2R_{12}^2-2R_{12}^3)(1-R_{13}-R_{23})\, m_{321}$$

$$= m_{321} - 2m_{411} + m_{42} + 2m_{51} - m_{33} = \sigma_3\sigma_2\sigma_1 - 2\sigma_4\sigma_1^2 + \sigma_4\sigma_2 + 2\sigma_5\sigma_1 - \sigma_3^2.$$

We introduce a family of polynomials $\{\Theta_\lambda\}$ indexed by k-strict partitions whose algebra is the same as the Schubert calculus in the stable cohomology ring of IG. Fix an integer $k \geq 0$ and consider a finite set of variables $y=(y_1,\ldots,y_k)$. For any $r \geq 0$, define ϑ_r by the equation

$$\prod_{i=1}^{\infty} \frac{1+x_it}{1-x_it} \prod_{j=1}^{k}(1+y_jt) = \sum_{r=0}^{\infty} \vartheta_r(x\,;y)t^r,$$

so that $\vartheta_r(x\,;y) = \sum_i q_{r-i}(x)e_i(y)$. We call $\Gamma^{(k)} := \mathbb{Z}[\vartheta_1,\vartheta_2,\ldots]$ the ring of *theta polynomials*. For any finite integer sequence α, let $\vartheta_\alpha = \prod_i \vartheta_{\alpha_i}$, and for any k-strict partition λ, define the theta polynomial

$$\Theta_\lambda := R^\lambda \vartheta_\lambda.$$

As a first application of Theorem 1, we obtain the next two results.

Theorem 2. *The Θ_λ, for λ k-strict, form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$. The algebra of theta polynomials agrees with the Schubert calculus on isotropic Grassmannians* $\mathrm{IG}(n-k,2n)$*, when n is sufficiently large.*

Theorem 3. *Let λ be a k-strict partition.*

a) *If $\lambda_i+\lambda_j \leq 2k+j-i$ for all $i<j$, then*

$$\Theta_\lambda(x\,;y) = \sum_{\mu\subset\lambda} S_\mu(x)s_{\lambda'/\mu'}(y), \quad \textit{where} \quad S_\mu(x) = \det(q_{\mu_i+j-i}(x)).$$

b) *If $\lambda_i+\lambda_j > 2k+j-i$ for all $i<j\leq \ell(\lambda)$, then*

$$\Theta_\lambda(x\,;y) = \sum_{\mu\subset\lambda:\,\mu\ \text{strict}} Q_\mu(x)s_{\mathcal{S}(\lambda/\mu)'}(y),$$

where $\mathcal{S}(\lambda/\mu)$ denotes a shifted skew diagram.

[1] As is customary, we slightly abuse the notation and consider that the raising operator R acts on the index α, and not on the monomial m_α itself.

It turns out that the coefficients of the polynomial $\Theta_\lambda(x\,;y)$ are always nonnegative integers, which have a combinatorial interpretation.

Theorem 4. *For any k-strict partition λ, the polynomial Θ_λ is a linear combination of products of Schur Q- and S-functions:*

$$\Theta_\lambda(x\,;y) = \sum_{\mu,\nu} e^\lambda_{\mu\nu} Q_\mu(x) s_{\nu'}(y).$$

Moreover, the coefficients $e^\lambda_{\mu\nu}$ are nonnegative integers, equal to the number of certain Kraśkiewicz tableaux [Kr] *of shape μ.*

In [T2], an approach to tableau formulas via raising operators is used to obtain a different expression for $\Theta_\lambda(x\,;y)$, which writes it as a sum of monomials $2^{n(U)}(xy)^U$ over all 'k-bitableaux' U of shape λ.

Our proof of Theorem 1 proceeds by showing directly that the raising operator expression $R^\lambda m_\lambda$ satisfies the Pieri rule for isotropic Grassmannians from [BKT1]. This is sufficient because the Pieri rule can be used recursively to show that a general Schubert class may be written as a polynomial in the special Schubert classes. The argument is challenging since the raising operator R^λ changes as the partition λ does, in contrast with the fixed operator expressions in (5). We remark that the equations corresponding to (5) for the Schur S- and Q-functions may be deduced from the formal identities

$$\text{(6)} \qquad \det(x_i^{\ell-j}) = \prod_{i<j}(x_i - x_j) \quad \text{and} \quad \text{Pfaffian}\left(\frac{x_i - x_j}{x_i + x_j}\right) = \prod_{i<j}\frac{x_i - x_j}{x_i + x_j}$$

due to Vandermonde and Schur, respectively (see e.g. [M, I.3 and III.8]).

Theorem 4 is proved by showing that the theta polynomial Θ_λ agrees with the type C Schubert polynomial of Billey and Haiman [BH] indexed by the corresponding Grassmannian element w_λ of the hyperoctahedral group. Our Giambelli formula may therefore be used to further understand these and related polynomials. For instance, it follows that the type C Stanley symmetric function $F_{w_\lambda}(x)$ of [BH, FK, L] is equal to $R^\lambda q_\lambda(x)$ (Corollary 6.2).

We have described the theory here in the symplectic case, but there are entirely analogous results for the odd orthogonal groups. In fact, for technical reasons, our proof of Theorem 1 is obtained in the setting of orthogonal type B. In a sequel to this paper, we will discuss the Giambelli formula for even orthogonal Grassmannians, which is more involved. We also obtain analogues of these Giambelli formulas for the quantum cohomology rings of symplectic and odd orthogonal Grassmannians; this application will appear in [BKT2].

This article is organized as follows. The proof of Theorem 1 occupies §1–§4. Section 5 introduces and studies theta polynomials, and contains our proofs of Theorems 2 and 3. Finally, in §6 we show that theta polynomials are Schubert polynomials for the hyperoctahedral group, and prove Theorem 4.

1. Preliminary Results

1.1. The Schubert varieties in $\mathrm{IG} = \mathrm{IG}(n-k, 2n)$ are indexed by k-strict partitions λ which are contained in an $(n-k)\times(n+k)$ rectangle; we denote the set of all such partitions by $\mathcal{P}(k,n)$. Consider the exact sequence of vector bundles over IG

$$0 \to \mathcal{S} \to E \to \mathcal{Q} \to 0,$$

where E denotes the trivial bundle of rank $2n$ and $\mathcal{S}$ is the tautological subbundle of rank $n-k$. The special Schubert class σ_p is equal to the Chern class $c_p(\mathcal{Q})$.

The symplectic form on E gives a pairing $\mathcal{S} \otimes \mathcal{Q} \to \mathcal{O}_{\mathrm{IG}}$, which in turn produces an injection $\mathcal{S} \hookrightarrow \mathcal{Q}^*$. For $r > k$ we therefore have

$$c_{2r}(\mathcal{Q} \oplus \mathcal{Q}^*) = c_{2r}(E/\mathcal{S} \oplus \mathcal{Q}^*) = c_{2r}(\mathcal{Q}^*/\mathcal{S}) = 0,$$

which implies that the relations

$$\sigma_r^2 + 2 \sum_{i=1}^{n+k-r} (-1)^i \sigma_{r+i}\sigma_{r-i} = 0 \quad \text{for } r > k \tag{7}$$

hold in $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$.

1.2. Let $\Delta^\circ = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j\}$ and define a partial order on Δ° by agreeing that $(i',j') \leq (i,j)$ if $i' \leq i$ and $j' \leq j$. We call a finite subset D of Δ° a *valid set of pairs* if it is an order ideal, i.e., $(i,j) \in D$ implies $(i',j') \in D$ for all $(i',j') \in \Delta^\circ$ with $(i',j') \leq (i,j)$. Given a k-strict partition λ and an integer $t \geq \ell(\lambda)$, we obtain a valid set of pairs $\mathcal{C}_t(\lambda)$ by

$$\mathcal{C}_t(\lambda) = \{(i,j) \in \Delta^\circ \mid \lambda_i + \lambda_j > 2k + j - i \;\text{ and }\; j \leq t\}.$$

Furthermore, we let $\mathcal{C}(\lambda) = \mathcal{C}_{\ell(\lambda)}(\lambda)$. It is easy to see that when $k > 0$, a set $D \subset \Delta^\circ$ is a valid set of pairs if and only if there exists a k-strict partition λ for which $\mathcal{C}(\lambda) = D$.

A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a vector of integers from the set $\mathbb{N} = \{0, 1, 2, \ldots\}$; we let $|\alpha| = \sum \alpha_i$. For λ any sequence of (possibly negative) integers, we say that λ has length ℓ if $\lambda_i = 0$ for all $i > \ell$ and $\ell \geq 0$ is the smallest number with this property. All integer sequences in this paper have finite length, and we will identify any integer sequence of length ℓ with the vector consisting of its first ℓ entries. In analogy with Young diagrams of partitions, we will say that a pair $[i,j]$ is a *box* of the integer sequence λ if $i \geq 1$ and $1 \leq j \leq \lambda_i$.

Any valid set of pairs D defines the raising operator

$$R^D = \prod_{i<j}(1 - R_{ij}) \prod_{i<j\,:\,(i,j)\in D} (1 + R_{ij})^{-1}.$$

Given a composition α and an integer $\ell > 0$, we denote by $m(D, \alpha, \ell)$ the number of non-zero coordinates α_i such that $(i, \ell) \in D$. We say that α is (D,ℓ)-*compatible* if $\alpha_i \in \{0, 1\}$ whenever $(i, \ell) \notin D$.

Definition 1.1. For any valid set of pairs D and any integer sequence λ of length ℓ we define a cohomology class $T_\lambda = T(D, \lambda)$ recursively as follows. Set $T_0 = 1$, $T_p = \sigma_p$, and $T_p = 0$ for $p < 0$. For an arbitrary integer sequence $\mu = (\mu_1, \ldots, \mu_{\ell-1})$ and $r \in \mathbb{Z}$, set

$$T_{\mu,r} = \sum_\alpha (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\mu+\alpha} T_{r-|\alpha|}\,, \tag{8}$$

where the sum is over all (D, ℓ)-compatible vectors $\alpha \in \mathbb{N}^{\ell-1}$.

The sum (8) is well defined because only finitely many of its summands are non-zero; we have $T_{\mu,r} = 0$ if $r < 0$. Notice that definition (8) of $T(D, \lambda)$ is equivalent to expanding the raising operator formula

$$R^D m_\lambda = \prod_{i<j<\ell}(1-R_{ij}) \prod_{i<j<\ell\,:\,(i,j)\in D}(1+R_{ij})^{-1} \prod_{i=1}^{\ell-1}(1-R_{i\ell}) \prod_{i\,:\,(i,\ell)\in D}(1+R_{i\ell})^{-1} m_{\mu,r}$$

after the last (i.e., the ℓ-th) entry of $\lambda = (\mu, r)$. Therefore $T_\lambda = R^D m_\lambda$.

1.3. If $D = \emptyset$ then for any integers r and s we have

$$T_{r,s} = T_r T_s - T_{r+1} T_{s-1}$$

and so $T_{r,r+1} = 0$, while more generally $T_{r,s} = -T_{s-1,r+1}$.

We claim that if $D \neq \emptyset$ and $r, s \in \mathbb{Z}$ are such that $r + s > 2k$, then $T_{s,r} = -T_{r,s}$; in particular $T_{r,r} = 0$ whenever $r > k$. Indeed, from the definition we obtain

$$T_{r,s} = \sigma_r\sigma_s - 2\,\sigma_{r+1}\sigma_{s-1} + 2\,\sigma_{r+2}\sigma_{s-2} - \cdots$$

and hence $T_{s,r} = -T_{r,s}$ whenever $r + s$ is odd. If $r + s = 2m > 2k$ is even, we see that

$$T_{r,s} + T_{s,r} = (-1)^{\frac{r-s}{2}}\, 2\,(\sigma_m^2 - 2\,\sigma_{m+1}\sigma_{m-1} + 2\,\sigma_{m+2}\sigma_{m-2} - \cdots) = 0 \qquad (9)$$

using the relations (7) in the cohomology ring of IG.

The previous observations are generalized in the next two lemmas.

Lemma 1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \ldots, \mu_\ell)$ be integer vectors. Assume that $(j, j+1) \notin D$ and that for each $h < j$, $(h, j) \notin D$ iff $(h, j+1) \notin D$. Then for any integers r and s we have*

$$T_{\lambda,r,s,\mu} = -T_{\lambda,s-1,r+1,\mu}\,.$$

In particular, $T_{\lambda,r,r+1,\mu} = 0$.

Proof. If $\mu = (\tau, t)$ has positive length, we set $\rho = (\lambda, r, s, \tau)$ and the identity follows by induction, because

$$T_{\rho,t} = \sum_\alpha (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\rho+\alpha} T_{t-|\alpha|}.$$

Therefore, we may assume that μ is empty. Set $\ell = j + 1$. Then we have

$$\begin{aligned} T_{\lambda,r,s} &= \sum_\alpha (-1)^{|\alpha|}\, 2^{m(D,\alpha,\ell)}\, T_{\lambda+\alpha,r} T_{s-|\alpha|} - \sum_\alpha (-1)^{|\alpha|}\, 2^{m(D,\alpha,\ell)}\, T_{\lambda+\alpha,r+1} T_{s-|\alpha|-1} \\ &= \sum_{\alpha,\beta} (-1)^{|\alpha|+|\beta|}\, 2^{m(D,\alpha,\ell)+m(D,\beta,\ell-1)}\, T_{\lambda+\alpha+\beta} T_{r-|\beta|} T_{s-|\alpha|} \\ &\quad - \sum_{\alpha,\beta} (-1)^{|\alpha|+|\beta|}\, 2^{m(D,\alpha,\ell)+m(D,\beta,\ell-1)}\, T_{\lambda+\alpha+\beta} T_{r+1-|\beta|} T_{s-1-|\alpha|} \end{aligned}$$

where the sums are over all (D, ℓ)-compatible sequences $\alpha \in \mathbb{N}^{j-1}$ and $(D, \ell - 1)$-compatible sequences $\beta \in \mathbb{N}^{j-1}$. The assumptions on D imply that these two sets of sequences coincide, and this proves the lemma. □

Lemma 1.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \ldots, \mu_\ell)$ be integer vectors, assume $(j, j+1) \in D$, and that for each $h > j + 1$, $(j, h) \in D$ iff $(j + 1, h) \in D$. If $r, s \in \mathbb{Z}$ are such that $r + s > 2k$, then we have*

$$T_{\lambda,r,s,\mu} = -T_{\lambda,s,r,\mu}\,.$$

In particular, $T_{\lambda,r,r,\mu} = 0$ for any $r > k$.

Proof. If $\mu = (\tau, t)$ has positive length, we set $\rho = (\lambda, r, s, \tau)$ and $\rho' = (\lambda, s, r, \tau)$, and the identity follows by induction, because

$$T_{\rho,t} = \sum_{\alpha} (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\rho+\alpha} T_{t-|\alpha|} = -\sum_{\alpha} (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\rho'+\alpha} T_{t-|\alpha|} = -T_{\rho',t}\,.$$

Thus we may assume that μ is empty. Set $\ell = j+1$, and note that $(h,h') \in D$ for all $h < h' \leq \ell$. If $m > 0$ is the least integer such that $2m \geq \ell$, we claim that $T_\rho = T_{\lambda,r,s}$ satisfies the relation

$$T_\rho = \sum_{i=2}^{2m} (-1)^i \, T_{\rho_1,\rho_i} \, T_{\rho_2,\ldots,\widehat{\rho_i},\ldots,\rho_{2m}}. \tag{10}$$

Equation (10) follows from the formal identity of raising operators

$$\prod_{1\leq h<h'\leq 2m} \frac{1-R_{hh'}}{1+R_{hh'}} = \sum_{i=2}^{2m} (-1)^i \, \frac{1-R_{1i}}{1+R_{1i}} \prod_{\substack{2\leq h<h'\leq 2m \\ h\neq i\neq h'}} \frac{1-R_{hh'}}{1+R_{hh'}},$$

which is equivalent the classical formula

$$\prod_{1\leq h<h'\leq 2m} \frac{x_h - x_{h'}}{x_h + x_{h'}} = \text{Pfaffian}\left(\frac{x_h - x_{h'}}{x_h + x_{h'}}\right)_{1\leq h,h'\leq 2m}$$

due to Schur [S, Sec. IX]. The proof is completed using induction, starting from the base case of $j = 1$, which was obtained in (9). □

During the above discussion the set D has remained fixed, but in subsequent arguments we will need to modify it. For this, we use a simple observation.

Lemma 1.3. *If $(i,j) \notin D$ and $D \cup (i,j)$ is a valid set of pairs, then*

$$T(D,\lambda) = T(D\cup(i,j),\lambda) + T(D\cup(i,j), R_{ij}\lambda).$$

Proof. The assertion follows immediately from the identity

$$1 - R_{ij} = \frac{1-R_{ij}}{1+R_{ij}} + \frac{1-R_{ij}}{1+R_{ij}}\, R_{ij}.$$ □

2. From $\mathrm{IG}(n-k,2n)$ to $\mathrm{OG}(n-k,2n+1)$

2.1. For each $k \geq 0$, the odd orthogonal Grassmannian $\mathrm{OG} = \mathrm{OG}(n-k, 2n+1)$ parametrizes the $(n-k)$-dimensional isotropic subspaces in $\mathbb{C}^{2n+1}$, equipped with a nondegenerate symmetric bilinear form. Our aim is to show that if λ is any k-strict partition, then $T(\mathcal{C}(\lambda),\lambda) = \sigma_\lambda$ in $\mathrm{H}^*(\mathrm{IG},\mathbb{Z})$. For technical reasons, we will use an isomorphism to transfer this relation to the cohomology ring of OG, and work with the latter space.

The Schubert varieties in OG are indexed by the same set of k-strict partitions $\mathcal{P}(k,n)$ as for $\mathrm{IG}(n-k,2n)$. Given a complete flag $F_\bullet$ of subspaces of $\mathbb{C}^{2n+1}$ such that $F_{n+i} = F_{n+1-i}^\perp$ for $1 \leq i \leq n+1$ and $\lambda \in \mathcal{P}(k,n)$, we define the codimension $|\lambda|$ Schubert variety

$$X_\lambda(F_\bullet) = \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma \cap F_{\overline{p}_j(\lambda)}) \geq j \ \ \forall\, 1 \leq j \leq \ell(\lambda)\}\,,$$

where

$$\overline{p}_j(\lambda) = n + k + 1 + j - \lambda_j - \#\{i \leq j : \lambda_i + \lambda_j > 2k + j - i\}. \tag{11}$$

Let $\tau_\lambda \in \mathrm{H}^{2|\lambda|}(\mathrm{OG},\mathbb{Z})$ be the cohomology class dual to the cycle given by $X_\lambda(F_\bullet)$.

For any $\lambda \in \mathcal{P}(k,n)$, let $\ell_k(\lambda)$ be the number of parts λ_i which are strictly greater than k. Let $\mathcal{Q}_{\mathrm{IG}}$ and $\mathcal{Q}_{\mathrm{OG}}$ be the universal quotient vector bundles over $\mathrm{IG}(n-k,2n)$ and $\mathrm{OG}(n-k,2n+1)$, respectively. It is known (see e.g. [BS, §3.1]) that the map which sends $\sigma_p = c_p(\mathcal{Q}_{\mathrm{IG}})$ to $c_p(\mathcal{Q}_{\mathrm{OG}})$ for all p extends to a ring isomorphism $\phi : \mathrm{H}^*(\mathrm{IG},\mathbb{Q}) \to \mathrm{H}^*(\mathrm{OG},\mathbb{Q})$. Moreover, we have $\phi(\sigma_\lambda) = 2^{\ell_k(\lambda)}\tau_\lambda$ for all $\lambda \in \mathcal{P}(k,n)$.

We let $c_p = c_p(\mathcal{Q}_{\mathrm{OG}})$. The Chern classes c_p are related to the special Schubert classes τ_p on OG by the equations

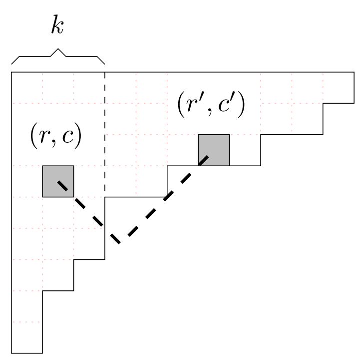
$$c_p = \begin{cases} \tau_p & \text{if } p \leq k, \\ 2\tau_p & \text{if } p > k. \end{cases}$$

Using the isomorphism ϕ, we can therefore describe the Giambelli formula for $\mathrm{OG}(n-k,2n+1)$ as follows. For any integer sequence α, set $m_\alpha = \prod_i c_{\alpha_i}$; then for every $\lambda \in \mathcal{P}(k,n)$, we have

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda \, m_\lambda \qquad (12)$$

in $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$.

2.2. For λ any k-strict partition, we say that the box $[r,c]$ in row r and column c of λ is *k-related* to the box $[r',c']$ if $|c-k-1|+r = |c'-k-1|+r'$. If $c \leq k < c'$, then this is equivalent to $c+c' = 2k+2+r-r'$. For example, in the partition displayed below, the grey box $[r,c]$ is k-related to $[r',c']$. The notion of k-related boxes makes sense also for boxes outside the Young diagram of λ.



Given two Young diagrams μ and ν with $\mu \subset \nu$, the skew diagram ν/μ is called a horizontal (resp. vertical) strip if it does not contain two boxes in the same column (resp. row). For any two k-strict partitions λ and μ, we write $\lambda \to \mu$ if μ may be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that

(1) if one of the first k columns of μ has the same number of boxes as the same column of λ, then the bottom box of this column is k-related to at most one box of $\mu \smallsetminus \lambda$; and

(2) if a column of μ has fewer boxes than the same column of λ, then the removed boxes and the bottom box of μ in this column must each be k-related to exactly one box of $\mu \smallsetminus \lambda$, and these boxes of $\mu \smallsetminus \lambda$ must all lie in the same row.

Equivalently, $\lambda \to \mu$ means that $\lambda_j - 1 \leq \mu_j \leq \lambda_{j-1}$ for each j, $\lambda_j \leq \mu_j$ when $\lambda_j > k$, and conditions (1) and (2) are true. Let $\mathbb{A}$ be the set of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ through $k+n$ which are not mentioned in (1) or (2), and define $\mathfrak{N}(\lambda,\mu)$ to be the number of connected components of $\mathbb{A}$. Here two boxes are connected if they share at least a vertex. In [BKT1, Theorem 2.1] we proved that the Pieri rule

$$c_p \cdot \tau_\lambda = \sum_{\lambda \to \mu,\, |\mu| = |\lambda| + p} 2^{\mathfrak{N}(\lambda,\mu)}\, \tau_\mu \tag{13}$$

holds, for any $p \in [1, n+k]$.

2.3. A comparison of (4) with (11) suggests modifying the definition of valid sets of pairs from §1 to include elements along the diagonal $\{(i,i) \mid i > 0\}$. This convention will make the formalism of our proof of Theorem 1 cleaner, and is in fact crucial in the corresponding proof of Giambelli for even orthogonal Grassmannians.

Set $\Delta = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq j\}$ with the same partial order as in §1.2, and define the notion of a valid set of pairs and the sets $\mathcal{C}_t(\lambda)$, $\mathcal{C}(\lambda)$ for a k-strict partition λ exactly as before, replacing Δ° by Δ. Thus $\mathcal{C}(\lambda)$ includes the pairs (i,i) such that $\lambda_i > k$. An *outer corner* of a valid set of pairs $D \subset \Delta$ is a pair $(i,j) \in \Delta \smallsetminus D$ such that $D \cup (i,j)$ is also a valid set of pairs. The *outside rim* ∂D of D is the set of pairs $(i,j) \in \Delta \smallsetminus D$ such that $i = 1$ or $(i-1, j-1) \in D$.

Lemma 2.1. *Let μ be a k-strict partition such that $\lambda \to \mu$. Then for any $t \geq \ell(\lambda)$, we have $\mathcal{C}_t(\lambda) \subset \mathcal{C}_{t+1}(\mu) \subset \mathcal{C}_t(\lambda) \cup \partial\, \mathcal{C}_t(\lambda)$.*

Proof. If $(i,j) \in \mathcal{C}_{t+1}(\mu)$, then $\lambda_{i-1} + \lambda_{j-1} \geq \mu_i + \mu_j > 2k + j - i$. This proves that $\mathcal{C}_{t+1}(\mu) \subset \mathcal{C}_t(\lambda) \cup \partial\mathcal{C}_t(\lambda)$. If there exists a pair $(i,j) \in \mathcal{C}_t(\lambda) \smallsetminus \mathcal{C}_{t+1}(\mu)$, then $\lambda_i + \lambda_j > 2k + j - i \geq \mu_i + \mu_j$, so we must have $\mu_i = \lambda_i$, $\mu_j = \lambda_j - 1$, and $\lambda_i + \lambda_j = 2k + 1 + j - i$. Condition (2) of §2.2 implies that some box $[h, c]$ of $\mu \smallsetminus \lambda$ is k-related to $[j, \lambda_j]$, and $[h, c-1]$ is also in $\mu \smallsetminus \lambda$ since this box is k-related to $[j-1, \lambda_j]$. The equality $\lambda_j + c = 2k + 2 + j - h$ implies that $(h,j) \in \mathcal{C}_{t+1}(\mu)$, and since $\mathcal{C}_{t+1}(\mu)$ is a valid set of pairs, we must have $h < i$. But we also obtain $\lambda_h < c - 1 = 2k + 1 + j - h - \lambda_j = \lambda_i + i - h$, contradicting the fact that λ is k-strict. This proves that $\mathcal{C}_t(\lambda) \subset \mathcal{C}_{t+1}(\mu)$. □

Definition 2.1. For any valid set of pairs $D \subset \Delta$ and any integer sequence λ we define the cohomology class $T(D,\lambda) \in \mathrm{H}^*(\mathrm{OG})$ by

$$T(D,\lambda) = 2^{-\#\{i \,|\, (i,i) \in D\}}\, \phi(T(D \cap \Delta^\circ, \lambda)),$$

where $T(D \cap \Delta^\circ, \lambda) \in \mathrm{H}^*(\mathrm{IG})$ is defined by (8).

To prove (12) and hence also establish Theorem 1, it suffices to show that if λ is a k-strict partition, the Pieri rule

$$c_p \cdot T(\mathcal{C}(\lambda), \lambda) = \sum_{\lambda \to \mu,\, |\mu| = |\lambda| + p} 2^{\mathfrak{N}(\lambda,\mu)}\, T(\mathcal{C}(\mu), \mu) \tag{14}$$

holds in $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z})$, for all p. To see this, write $\mu \succ \lambda$ if μ strictly dominates λ, i.e., $\mu \neq \lambda$ and $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$ for each $i \geq 1$. We deduce from (13) and (14) that

$$2^{\ell_k(\lambda)}\, \tau_\lambda + \sum_{\mu \succ \lambda} a_\mu\, \tau_\mu = c_{\lambda_1} \cdots c_{\lambda_\ell} = 2^{\ell_k(\lambda)}\, T(\mathcal{C}(\lambda), \lambda) + \sum_{\mu \succ \lambda} a_\mu\, T(\mathcal{C}(\mu), \mu),$$

for some constants $a_\mu \in \mathbb{Z}$. The proof now follows by induction.

Observe that Lemmas 1.1, 1.2, and 1.3 carry over verbatim to our current setting where $D \subset \Delta$. These lemmas are the main properties of the cohomology classes $T(D, \lambda)$ that we use, and as such constitute the technical core of our proof of Theorem 1. But the non-trivial scheme that puts them to work together is an algorithm with a substitution rule; this is explained in the next section.

3. The Substitution Rule

3.1. Throughout the next two sections we fix $p > 0$, the k-strict partition λ of length ℓ, and choose n sufficiently large so that we can ignore it in the sequel. Set $\mathcal{C} = \mathcal{C}(\lambda)$. For any $d \geq 1$ define the raising operator R^λ_d by

$$R^\lambda_d = \prod_{1\leq i<j\leq d} (1 - R_{ij}) \prod_{i<j\,:\,(i,j)\in\mathcal{C}} (1 + R_{ij})^{-1}.$$

We compute that

$$c_p \cdot T(\mathcal{C}, \lambda) = c_p \cdot 2^{-\ell_k(\lambda)} R^\lambda_\ell \, m_\lambda = 2^{-\ell_k(\lambda)} R^\lambda_{\ell+1} \cdot \prod_{i=1}^{\ell} (1 - R_{i,\ell+1})^{-1} \, m_{\lambda,p}$$

$$= 2^{-\ell_k(\lambda)} R^\lambda_{\ell+1} \cdot \prod_{i=1}^{\ell} (1 + R_{i,\ell+1} + R^2_{i,\ell+1} + \cdots) \, m_{\lambda,p}$$

and therefore

$$c_p \cdot T(\mathcal{C}, \lambda) = \sum_{\nu\in\mathcal{N}} T(\mathcal{C}, \nu), \tag{15}$$

where $\mathcal{N} = \mathcal{N}(\lambda, p)$ is the set of all compositions $\nu \geq \lambda$ such that $|\nu| = |\lambda| + p$ and $\nu_j = 0$ for $j > \ell + 1$. Our strategy for proving Theorem 1 is to show that the right hand side of equation (15) is equal to the right hand side of the Pieri rule (14).

3.2. The following objects which will used as book keeping tools in a delicate process of rewriting the right hand side of (15).

Definition 3.1. A *valid 4-tuple* of *level* h is a 4-tuple $\psi = (D, \mu, S, h)$, such that h is an integer with $0 \leq h \leq \ell+1$, D is a valid set of pairs containing $\mathcal{C}$, all pairs (i, j) in D satisfy $j \leq \ell+1$, S is a subset of $D \smallsetminus \mathcal{C}$, and μ is an integer sequence of length at most $\ell + 1$. The evaluation of ψ is defined by $\mathrm{ev}(\psi) = T(D, \mu) \in H^*(\mathrm{OG}, \mathbb{Z})$.

All valid 4-tuples encountered in this paper will also satisfy that $D \subset \mathcal{C} \cup \partial\mathcal{C}$ (see Lemma 4.1), but for technical reasons we do not require this in the definition. We will represent the set Δ as the positions on or above the main diagonal of a matrix, and the various sets of pairs D as sets of entries in this matrix. In Figure 1 the white dots represent a set of pairs $\mathcal{C}$ and the grey dots are a subset of the outside rim of $\mathcal{C}$. The union of the white and grey dots form the set D in a typical valid 4-tuple (D, μ, S, h).

In the following we set $\mu_0 = \infty$ whenever μ is an integer sequence.

Definition 3.2. For any $y \in \mathbb{Z}$ we let $r(y)$ denote the largest integer such that $r(y) \leq \ell + 1$ and $\lambda_{r(y)-1} > 2k + r(y) - y$.

Since $\lambda_0 = \infty$ we have $r(y) \geq 1$. Notice that for $(i, j) \in \Delta$ and $j \leq \ell$ we have $(i, j) \in \mathcal{C} \Leftrightarrow \lambda_j > 2k + j - i - \lambda_i \Leftrightarrow j < r(i + \lambda_i + 1)$. This gives the relation

$$\mathcal{C} = \{(i, j) \in \Delta \mid \text{ and } j < r(i + \lambda_i + 1)\}\,. \tag{16}$$

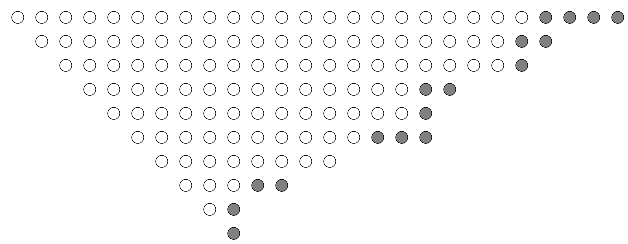


FIGURE 1. A valid set of pairs $\mathcal{C}$ (white dots) and a subset of $\partial\mathcal{C}$ (grey dots).

The function $r(y)$ is also connected to the notion of k-relatedness of boxes. Assume that some box $[i,c]$ with $c > k$ is k-related to a box $[j,d]$ in the first k columns of λ. Then $\lambda_j \geq d = 2k+2+j-i-c$, which implies that $j < r(i+c)$. Furthermore, if $[j+1,d+1] \notin \lambda$ then $r(i+c) = j+1$. In other words, the south-east most box of the first k columns of λ that is k-related to $[i,c]$ is located in row $r(i+c)-1$.

Let $m \geq 1$ be minimal such that $\lambda_m \leq k$; we call m the *middle* row of λ. Notice that m is the smallest positive integer for which $(m,m) \notin \mathcal{C}$, so $m = 10$ in Figure 1.

Definition 3.3. Let $h \in \mathbb{N}$ satisfy $1 \leq h \leq m$ and let μ be an integer sequence.

(a) We define $b_h = r(h+\lambda_h+1)$ and $g_h = b_{h-1}$. By convention we set $g_1 = \ell+1$.

(b) Set $R(\mu) = \{[i,c] \in \mu \smallsetminus \lambda \mid c > k \text{ and } \mu_{r(i+c)} \leq 2k + r(i+c) - i - c\}$.

(c) Assume that $h \geq 2$ and $\mu_h \geq \lambda_{h-1}$. If $[h, \lambda_{h-1}] \in R(\mu)$ then set $e_h(\mu) = \lambda_{h-1}$. Otherwise choose $e_h(\mu) > \max\{k, \lambda_h\}$ minimal such that $[h,c] \notin R(\mu)$ for $e_h(\mu) \leq c \leq \lambda_{h-1}$. Finally, set $f_h(\mu) = r(h + e_h(\mu))$.

Notice that for $h < m$ we have $b_h = \min\{j \geq m \mid (h,j) \notin \mathcal{C}\}$. In the definition of $R(\mu)$, suppose that some box $[i,c] \in \mu \smallsetminus \lambda$ with $c > k$ is k-related to a box $[j,d]$ in the first k columns of λ, such that $[j+1,d+1] \notin \lambda$. Then we have $r(i+c) = j+1$ and $d = 2k+2+j-i-c$. It follows that $[i,c] \in R(\mu)$ if and only if $\mu_{j+1} < d$. In particular, if μ is a k-strict partition such that $\lambda \to \mu$, then the set $\mathbb{A}$ from §2.2 consists of the boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ and higher which are not in $R(\mu)$. Notice also that since $h + \lambda_h + 1 \leq h + e_h(\mu) \leq h + \lambda_{h-1}$ and $r(y)$ is an increasing function of y, we always have

$$b_h \leq f_h(\mu) \leq g_h \,. \tag{17}$$

Furthermore, if $[h, \lambda_{h-1}] \in R(\mu)$, then $f_h(\mu) = g_h$.

If we are given a fixed valid 4-tuple (D, μ, S, h) with $1 \leq h \leq m$, we will use the shorthand notation $b = b_h$, $g = g_h$, $R = R(\mu)$, $e = e_h(\mu)$, and $f = f_h(\mu)$; the values e and f will be used only when $\mu_h \geq \lambda_{h-1}$. The integers b and g are illustrated in Figure 2.

Example 3.1. Suppose $k = 3$ and $\lambda = (9,7,3,2,1,1)$, so that

$$\mathcal{C}(\lambda) = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4)\}.$$

Consider a valid 4-tuple (D, μ, S, h) with $\mu = (11,12,7,2,2)$. Figure 3 illustrates λ and μ, with the boxes in $\mu \smallsetminus \lambda$ shaded, and the boxes in R marked. Note that there is one box in $\lambda \smallsetminus \mu$. If $h = 2$ then we obtain $(e,f,g) = (8,5,5)$, while if $h = 3$ we get $(e,f,g) = (6,4,5)$.

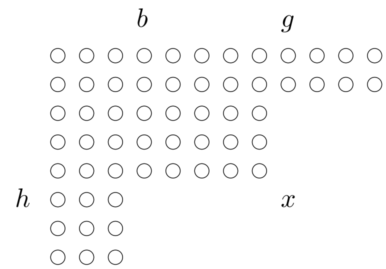


FIGURE 2. The set $\mathcal{C}$ near the pair $x = (h, g) \in \partial\mathcal{C}$.

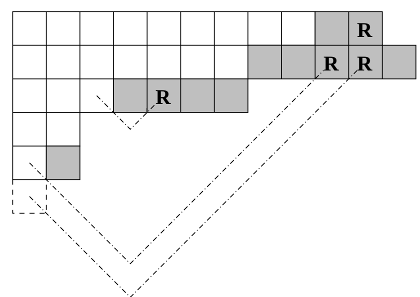


FIGURE 3. The shapes λ and μ, with $\mu \smallsetminus \lambda$ shaded.

The precise value of f will play a crucial role in our proof that the Pieri terms in (14) appear in (15) with the correct multiplicities. For example, it is part of the following definition of a condition X, that will be used to identify undesired valid 4-tuples.

Definition 3.4. Let $(i, j) \in \Delta$ be arbitrary. We define two conditions $\mathrm{W}(i, j)$ and X on a valid 4-tuple (D, μ, S, h) as follows.

$$\mathrm{W}(i, j) \ : \ \mu_i + \mu_j > 2k + j - i\,.$$

Condition X is true if and only if $(h, h) \in D$ and

$$\mu_h \geq \mu_{h-1} \text{ or } \mu_h > \lambda_{h-1} \text{ or } (\mu_h = \lambda_{h-1} \text{ and } (h, f) \notin S)\,.$$

3.3. Initially, we define the set $\Psi = \{(\mathcal{C}, \nu, \emptyset, \ell{+}1) \mid \nu \in \mathcal{N}(\lambda, p)\}$; thus $\sum_{\psi\in\Psi} \mathrm{ev}(\psi)$ agrees with the right hand side of (15). We then apply an *algorithm* which will change this set by replacing some 4-tuples with one or two new valid 4-tuples. The algorithm applies the *substitution rule* described below to each element (D, μ, S, h) in Ψ of level $h \geq 1$. If the substitution rule results in a REPLACE statement, then the set Ψ is changed accordingly; otherwise the substitution rule results in a STOP statement, in which case the 4-tuple (D, μ, S, h) is left untouched. These substitutions are iterated until no further elements in Ψ can be REPLACED, i.e., until the substitution rule results in a STOP statement when applied to any 4-tuple in Ψ with $h \geq 1$.

Substitution Rule

Suppose that $h \geq 1$. Assume first that $(h,h) \notin D$. If

(i) there is an outer corner (i,h) of D such that $\mathrm{W}(i,h)$ holds

then REPLACE (D,μ,S,h) with

$$(D \cup (i,h), \mu, S, h) \ \text{ and } \ (D \cup (i,h), R_{ih}\mu, S \cup (i,h), h).$$

Otherwise, if

(ii) D has no outer corner in column h and $\mu_h > \lambda_{h-1}$,

then STOP.

Assume now that $(h,h) \in D$. If

(iii) there is an outer corner (h,j) of D with $j \leq \ell+1$ such that $\mathrm{W}(h,j)$ holds,

then REPLACE (D,μ,S,h) with

$$\begin{cases} (D \cup (h,j), \mu, S, h) \ \text{ and } \ (D \cup (h,j), R_{hj}\mu, S \cup (h,j), h) & \text{if } \mu_j \leq \mu_{j-1}, \\ (D \cup (h,j), R_{hj}\mu, S \cup (h,j), h) & \text{if } \mu_j > \mu_{j-1}. \end{cases}$$

Otherwise, if

(iv) $\mathrm{W}(h,g)$ or X holds, and D has an outer corner (i,g) with $i \leq h$,

then REPLACE (D,μ,S,h) with

$$(D \cup (i,g), \mu, S, h) \ \text{ and } \ (D \cup (i,g), R_{ig}\mu, S \cup (i,g), h).$$

Otherwise, if

(v) X holds,

then STOP.

If none of the above conditions hold, REPLACE (D,μ,S,h) with $(D,\mu,S,h-1)$.

Since the set of pairs D is not allowed to grow beyond column $\ell+1$, the algorithm will terminate after a finite number of steps. Whenever a 4-tuple $\psi = (D,\mu,S,h)$ is replaced by one or two 4-tuples ψ_i, we refer to ψ as the *parent* term and the ψ_i are its *children*. Notice that we can recover the initial 4-tuple $\psi_0 = (\mathcal{C},\nu,\emptyset,\ell+1)$ that gave rise to ψ by the equation $\nu = \prod_{(i,j)\in S} L_{ij}\mu$. Here L_{ij} denotes the lowering operator which is the inverse of R_{ij}. Furthermore, the sequence of 4-tuples leading from ψ_0 to ψ is uniquely determined by ψ. In particular, no 4-tuple can be produced multiple times.

If a 4-tuple $\psi \in \Psi$ is REPLACED by two 4-tuples ψ_1 and ψ_2, we deduce from Lemma 1.3 that $\mathrm{ev}(\psi) = \mathrm{ev}(\psi_1) + \mathrm{ev}(\psi_2)$. Moreover, if condition **(iii)** holds for $\psi = (D,\mu,S,h)$ and this 4-tuple is REPLACED by the single 4-tuple $\psi' = (D \cup (h,j), R_{hj}\mu, S \cup (h,j), h)$, then Lemmas 1.1 and 1.3 imply that $\mathrm{ev}(\psi) = \mathrm{ev}(\psi')$. Indeed, it follows from Corollary 4.1 below that $\mu_{j-1} = \mu_j - 1$ and $D \cup (h,j)$ has no outer corner in column j, so Lemma 1.1 shows that $\mathrm{ev}(D \cup (h,j), \mu, S, h) = 0$.

When the algorithm terminates, let Ψ_0 denote the set of 4-tuples $(D,\mu,S,0)$ in Ψ and let $\Psi_1 = \Psi \smallsetminus \Psi_0$. It follows that

$$\sum_{\nu \in N} T(\mathcal{C},\nu) = \sum_{\psi \in \Psi_0} \mathrm{ev}(\psi) + \sum_{\psi \in \Psi_1} \mathrm{ev}(\psi).$$

In the next section, we will prove the following two claims.

Claim 1. For each 4-tuple $\psi = (D, \mu, S, 0)$ in Ψ_0 with $\mu_{\ell+1} \geq 0$, μ is a k-strict partition with $\lambda \to \mu$ and $\mathrm{ev}(\psi) = T(\mathcal{C}(\mu), \mu)$. Furthermore, for each such partition μ, there are exactly $2^{\mathfrak{N}(\lambda,\mu)}$ such 4-tuples ψ, in accordance with the Pieri rule.

Claim 2. There exists an involution $\iota : \Psi_1 \to \Psi_1$ of the form $\iota(D, \mu, S, h) = (D, \mu', S', h)$ such that $\mathrm{ev}(\psi) + \mathrm{ev}(\iota(\psi)) = 0$, for every $\psi \in \Psi_1$.

We remark that the 4-tuples $\psi \in \Psi_0$ with $\mu_{\ell+1} < 0$ evaluate to zero trivially, by Definition 1.1. The two claims therefore suffice to prove the Pieri rule (14).

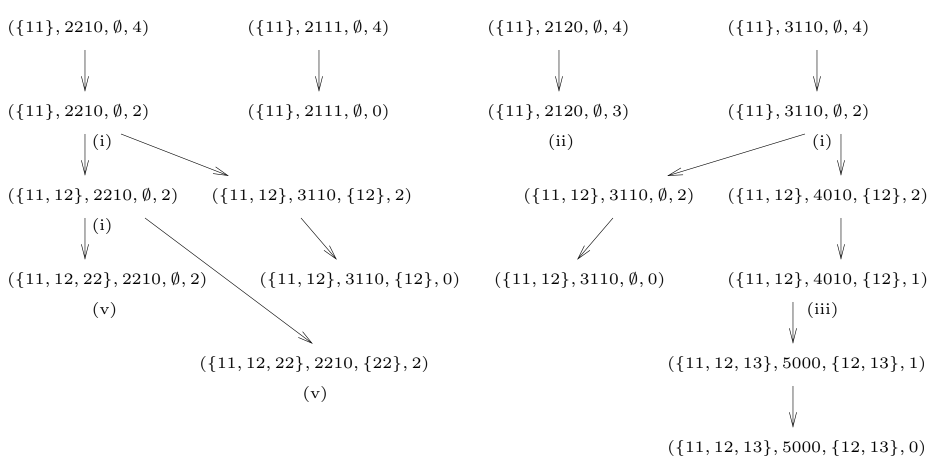
Definition 3.5. Let **(x)** be one of the conditions **(i)**–**(v)** of the Substitution Rule. We say that a 4-tuple $\psi = (D, \mu, S, h)$ *meets* condition **(x)** if ψ occurs in the algorithm, reaches condition **(x)** in the Substitution Rule, and condition **(x)** is satisfied. We say that a 4-tuple ψ *survives the algorithm* if at least one of its successors lies in Ψ_0.

For each initial 4-tuple $\psi_0 = (\mathcal{C}, \nu, \emptyset, \ell{+}1)$ of the sum (15), the algorithm produces a tree of 4-tuples with root node given by ψ_0. If the Substitution Rule REPLACES a 4-tuple ψ by one or two other 4-tuples ψ_i, we have a branch in the tree from ψ to the ψ_i. The leaves of the tree are exactly the 4-tuples with $h = 0$ or where the Substitution Rule STOPS. The fate of all the terms of the sum (15) is encoded by the collection of all the trees with root nodes $(\mathcal{C}, \nu, \emptyset, \ell + 1)$ for $\nu \in \mathcal{N}(\lambda, p)$. This collection will be called the *substitution forest*; the sum of the cohomology classes represented by the roots of the substitution forest is equal to the sum of classes given by the leaves.

Example 3.2. We discuss an example of the substitution forest in detail. Consider the Grassmannian $\mathrm{OG}(n-1, 2n+1)$ for $n \geq 5$, and the Pieri product

$$c_1 \cdot \tau_{2,1,1} = \tau_{2,1,1,1} + 2\,\tau_{3,1,1} + \tau_5.$$

For simplicity, we will omit the commas in our notation for compositions and pairs. Thus $\lambda = 211$, $k = p = 1$, and we have $\mathcal{C}(\lambda) = \{11\}$ and $\mathcal{N}(\lambda, p) = \{2111, 2120, 2210, 3110\}$. The substitution forest is pictured below, except we have omitted those nodes (D, μ, S, h) which have $(D, \mu, S, h{+}1)$ as parent and $(D, \mu, S, h{-}1)$ as child.



Observe that the root $(\{11\}, 2120, \emptyset, 4)$ is the only initial 4-tuple that does not survive the algorithm. We have $\Psi_0 = \{\ (\{11\}, 2111, \emptyset, 0),\ (\{11, 12\}, 3110, \{12\}, 0),\ (\{11, 12\}, 3110, \emptyset, 0),\ (\{11, 12, 13\}, 5000, \{12, 13\}, 0)\ \}$, which corresponds exactly to the terms in the Pieri product $c_1 \cdot \tau_{211}$. Furthermore, each 4-tuple in the set $\Psi_1 = \{\ (\{11, 12, 22\}, 2210, \emptyset, 2),\ (\{11, 12, 22\}, 2210, \{22\}, 2),\ (\{11\}, 2120, \emptyset, 3)\ \}$ evaluates to zero in the cohomology ring of OG.

4. Proof of Theorem 1

4.1. Recall the fixed choices of p, λ, ℓ, $\mathcal{C}$, and m from §3.1. In §4.1 through §4.3 we furthermore let $\psi = (D, \mu, S, h)$ denote a 4-tuple which occurs at some step in the algorithm, i.e., a node of the substitution forest. The symbols D, μ, S, h will refer to components of the 4-tuple ψ. We will occasionally work with more than one valid 4-tuple. If (D', μ', S', h') is an additional 4-tuple, then the sets and values that Definition 3.3 associates to this 4-tuple will be called R', e', f', and g'.

The algorithm has two phases. A 4-tuple ψ is in Phase 1 if $(h, h) \notin D$, and in Phase 2 if $(h, h) \in D$. The level h is always used to index an entry of the integer sequence μ in ψ; it begins at $h = \ell+1$ and decreases as the 4-tuple proceeds through the algorithm. In Phase 1 we have $h \geq m$, while $h \leq m$ in Phase 2. Throughout the algorithm we have $i \leq m \leq j$ for each $(i, j) \in S$, so μ is obtained from the initial composition ν by removing boxes from rows weakly below the middle row of λ and adding them to rows weakly above the middle row.

The set D is initially equal to $\mathcal{C}$ and grows when REPLACE statements are encountered. Lemma 4.1 below shows that all pairs added to D come from the outer rim $\partial\mathcal{C}$. In Phase 1, pairs are added by rule **(i)** to column h, so as the level h decreases from $\ell + 1$ to m, these pairs are added along vertical columns of $\partial\mathcal{C}$, proceeding from top (row 1) to bottom (row m) and right to left. In Phase 2, the set D mainly grows when rule **(iii)** adds pairs to row h, in which case the pairs are added in horizontal rows of $\partial\mathcal{C}$, from left to right and bottom to top. In some cases rule **(iv)** will add extra pairs (i, g) to D, where $i \leq h$. Lemma 4.5 implies that if ψ meets **(iv)**, then it will not survive the algorithm, and only pairs from column g of $\partial\mathcal{C}$ can be added to its successors. In particular, all 4-tuples in Ψ_0 are produced from the the initial 4-tuples by applications of rules **(i)** and **(iii)**.

Our proof of Theorem 1 occupies the remainder of this section. In §4.2 we prove some properties satisfied by 4-tuples that occur in the algorithm. Additional properties for 4-tuples in Ψ_0 are proved in §4.3. The proof of Claim 1 is then given in §4.4, while Claim 2 is justified in §4.5.

4.2. We prove some lemmas that reveal what can happen to the 4-tuple $\psi = (D, \mu, S, h)$ during the algorithm.

Lemma 4.1. *We have $D \subset \mathcal{C} \cup \partial\mathcal{C}$.*

Proof. It is enough to show that if the substitution rule adds the pair (i, j) to D, then $(i, j) \in \partial\mathcal{C}$. Notice first that $i \leq h \leq j$. If $(i, j) \notin \partial\mathcal{C}$, then $i > 1$ and $(i-1, j-1) \notin \mathcal{C}$. Let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ such that $(i-1, j) \notin D'$. Then ψ' meets rule **(i)**, **(iii)**, or **(iv)**, which adds the pair $(i-1, j)$ to D'. Since the pair $(i-1, j-1) \in D' \smallsetminus \mathcal{C}$ was added to a predecessor of ψ' of level smaller than j, it follows that $i \leq h \leq h' \leq j-1$, so ψ' does not meet **(i)** or **(iii)**. But ψ' also does not meet **(iv)** because $g' \leq j-1$, a contradiction. $\square$

Lemma 4.2. *Suppose that $h \leq m$ and $\mu_h \geq \lambda_{h-1}$. Suppose $\overline{\mu}$ is an integer sequence such that $\overline{\mu}_h \geq \lambda_{h-1}$ and $\overline{\mu}_j = \mu_j$ for $\max(m, h+1) \leq j \leq g$. Then $[h,c] \in R$ if and only if $[h,c] \in R(\overline{\mu})$ for all $c \leq \lambda_{h-1}$. In particular, we have $e = e_h(\overline{\mu})$ and $f = f_h(\overline{\mu})$.*

Proof. Let $[h,c] \in \mu \smallsetminus \lambda$ satisfy $k < c \leq \lambda_{h-1}$, and set $j = r(h+c)$. Since k-strictness of λ implies that $k+m+1 \leq h+c \leq h+\lambda_{h-1}$, we obtain $m \leq j \leq g$ and hence $\mu_j = \overline{\mu}_j$ provided that $j > h$. It follows that $[h,c] \in R$ if and only if $[h,c] \in R(\overline{\mu})$, as required. □

Lemma 4.3. *If $(h,h) \in D$, $\mu_h \geq \lambda_{h-1}$ and $\lambda_{h-1} + \mu_g \leq 2k + g - h$, then $f = g$.*

Proof. Since $h \leq m$ we have $\lambda_{h-1} > k$ and $[h, \lambda_{h-1}] \in \mu \smallsetminus \lambda$. Since $g = r(h+\lambda_{h-1})$, the inequality $\mu_g \leq 2k + g - h - \lambda_{h-1}$ shows that $[h, \lambda_{h-1}] \in R$. This implies that $e = \lambda_{h-1}$ and $f = r(h + \lambda_{h-1}) = g$, as required. □

We next make some observations concerning rule **(iv)** and condition X.

Lemma 4.4. *If condition* X *holds for ψ, then* X *also holds for the children of ψ. In particular, ψ does not survive the algorithm, and all its successors have level h.*

Proof. Assume that $\psi = (D, \mu, S, h)$ satisfies condition X and let $\psi' = (D', \mu', S', h)$ be a child of ψ. If $S' = S$ then $\mu' = \mu$ and X also holds for ψ', so we may assume that $S' \smallsetminus S = \{(i,j)\}$ and $\mu' = R_{ij}\mu$, where $i \leq h$. We can also assume that $\lambda_{h-1} = \mu_h = \mu'_h < \mu'_{h-1}$. Since $(h,h) \in D$ and $(i,j) \notin D$, we get $i < h < j$. In particular, ψ meets **(iv)** and $j = g$. Let $\overline{\psi} = (\overline{D}, \overline{\mu}, \overline{S}, g)$ be the most recent predecessor of ψ of level g, and let (a,g) be the outer corner of $\overline{D}$ in column g. Then $a \leq h-1$, and W(a,g) fails for $\overline{\psi}$ since it does not meet **(i)**. Using this and k-strictness of λ, we obtain $\lambda_{h-1} + \mu'_g < \lambda_{h-1} + \mu_g \leq (\lambda_a + a - h + 1) + \overline{\mu}_g \leq \overline{\mu}_a + \overline{\mu}_g + 1 + a - h \leq 2k + 1 + g - h$. Lemma 4.3 now implies that $f' = g$. We conclude that ψ' satisfies condition X as $\mu'_h = \lambda_{h-1}$ and $(h, f') = (h,g) \notin S'$. □

Lemma 4.5. *Suppose that ψ meets* **(iv)**, *and let (a,g) be the outer corner of $\mathcal{C}$ in column g. Then ψ does not survive the algorithm, and all of its successors $\psi' = (D', \mu', S', h')$ have level $h' > a$.*

Proof. By Lemma 4.4 we may assume that X fails and W(h,g) holds for ψ. In particular, $\mu_{h-1} > \mu_h$. Assume that $\mu_a > \mu_{a+1} > \cdots > \mu_{h-1} > \mu_h$. Since W$(h,g)$ holds, this implies that W(i,g) is true for $a \leq i < h$. Moreover, these latter weight conditions are true for all predecessors of ψ. It follows that all pairs (i,g) for $a \leq i < h$ were added to D by **(i)** during Phase 1 of the algorithm. But then (h,g) must be an outer corner of D, so ψ meets **(iii)**, a contradiction.

We therefore have $\mu_{i-1} \leq \mu_i$ for some integer i with $a < i < h$. We claim that any successor $\psi' = (D', \mu', S', i)$ of ψ of level i satisfies condition X. Otherwise, $\mu'_i < \mu'_{i-1}$ and $\mu'_i \leq \lambda_{i-1}$, and since $\mu_i \leq \mu'_i$ and $\lambda_{i-1} \leq \mu_{i-1}$, this implies that $\mu_i = \mu'_i = \lambda_{i-1} = \mu_{i-1}$. Since $\mu'_{i-1} > \mu_{i-1}$, it follows that $(i-1, g) \in D' \smallsetminus D$. Therefore $(i,g) \notin S$, and since $\mu'_i = \mu_i$, we deduce that $(i,g) \notin S'$. But (17) implies that $f' = g$, so X holds for ψ' anyway. We conclude that ψ will not survive. □

We now prove some results that will later be used to show that surviving 4-tuples $\psi = (D, \mu, S, 0) \in \Psi_0$ satisfy $\lambda \to \mu$.

Lemma 4.6. *If $j > h$ and $(j,j) \notin D$, then $\mu_j \leq \lambda_{j-1}$.*

Proof. Assume that $\mu_j > \lambda_{j-1}$ and let $\psi' = (D', \mu', S', j)$ be the most recent predecessor of ψ of level j. Then $\mu'_j \geq \mu_j$, and since ψ' does not meet **(ii)**, it follows that D' has an outer corner (i,j) in column j. If $i < j$, then since $(i, j-1) \in \mathcal{C}$ we obtain $\mu'_j + \mu'_i > \lambda_{j-1} + \lambda_i > 2k + (j-1) - i$, and otherwise we have $i = j = m$ and $\mu'_j > \lambda_{m-1} > k$. In both cases ψ' satisfies $\mathrm{W}(i,j)$. But then ψ' meets **(i)** and is not the most recent predecessor of ψ of level j, a contradiction. □

Lemma 4.7. *Let $\psi = (D, \mu, S, h)$ and let $j \leq \ell$ be a positive integer.*
(a) *If $h \geq 1$ and $(h,j) \notin \mathcal{C}$ and $(h+1, j) \in D$, then we have $\mu_j \geq \lambda_j$.*
(b) *If $h \leq 1$ or $(h-1, j) \in \mathcal{C}$, then we have $\mu_j \geq \lambda_j - 1$. Moreover, if $\mu_j = \lambda_j - 1$, then $D \smallsetminus \mathcal{C}$ contains exactly one pair in column j, and this pair is also in S.*

Proof. Suppose that $\mu_j < \lambda_j$ and choose $i > h$ maximal such that $(i,j) \in D$. Let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ with $(i,j) \notin D'$. Then ψ' meets rule **(i)**, **(iii)**, or **(iv)**, which adds the pair (i,j) to D'. Let $\overline{\psi} = (D' \cup (i,j), \overline{\mu}, \overline{S}, h')$ be the child of ψ' that is a predecessor of ψ. Notice that $\mu'_t \leq \overline{\mu}_t \leq \lambda_{t-1}$ for all integers t such that $h < t \leq h'$ and $(t,t) \in D' \cup (i,j)$, since otherwise condition X holds for every successor of $\overline{\psi}$ of level t. We also have $\mu'_j \leq \lambda_j$, and if $\mu'_j = \lambda_j$ then $(i,j) \in \overline{S}$, $i < j$, $\overline{\mu}_i > \mu'_i$, and $\overline{\mu}_j < \mu'_j$.

If ψ' meets **(i)**, then $h' = j$. Since $(i-1, j) \notin \mathcal{C}$ we have $\mu'_i + \mu'_j \leq \lambda_{i-1} + \lambda_j \leq 2k + j - i + 1$. As $\mathrm{W}(i,j)$ holds for ψ', it follows that $\mu'_i = \lambda_{i-1}$ and $\mu'_j = \lambda_j$. But this implies that $\overline{\mu}_i > \lambda_{i-1}$, a contradiction.

Therefore ψ' meets **(iii)** with $h' = i$, or it meets **(iv)** with $h' \geq i$. In either case we have $g' = j$, and since ψ' does not satisfy condition X, it must satisfy $\mathrm{W}(h', j)$. Since $(h'-1, j) \notin \mathcal{C}$ and thus $\mu'_{h'} + \mu'_j \leq \lambda_{h'-1} + \lambda_j \leq 2k + j - h' + 1$, it follows that $\mu'_{h'} = \lambda_{h'-1}$ and $\overline{\mu}_j < \mu'_j = \lambda_j$. We obtain $\lambda_{h'-1} + \overline{\mu}_j \leq 2k + j - h'$, so Lemma 4.3 shows that $\overline{f} = j$. Since $\overline{\mu}_i > \mu'_i$, we must also have $i < h'$, so $(h', \overline{f}) \notin \overline{S}$ and $\overline{\psi}$ satisfies condition X. This contradiction completes the proof of part (a).

If $\mu_j \leq \lambda_j - 2$, then $D \smallsetminus \mathcal{C}$ contains at least two pairs in column j, say $(a+1, j)$ and (a, j), and the assumptions in (b) imply that $a \geq h$. Let $\psi' = (D', \mu', S', a)$ be the most recent predecessor of ψ of level a. Part (a) applied to ψ' implies that $\mu'_j \geq \lambda_j$, a contradiction since $\mu'_j = \mu_j$. □

Corollary 4.1. *Assume that ψ meets* **(iii)** *and let (h,j) be the outer corner of D in row h. If $\mu_j > \mu_{j-1}$, then $\mu_{j-1} = \mu_j - 1$ and $D \cup (h,j)$ has no outer corner in column j.*

Proof. Lemma 4.6 implies that $\mu_{j-1} < \mu_j \leq \lambda_{j-1}$. Since $(h-1, j-1) \in \mathcal{C}$, it follows from Lemma 4.7(b) that $\mu_j = \lambda_{j-1} = \mu_{j-1} + 1$ and $D \smallsetminus \mathcal{C}$ contains a unique pair $(i, j-1)$ in column $j-1$. Since $i < j-1$, it is enough to show that $i = h$. Lemma 4.5 implies that $(i, j-1)$ was added by **(i)** or **(iii)**, so ψ satisfies $\mathrm{W}(i, j-1)$, and we obtain $\mu_i + \mu_j = \mu_i + \mu_{j-1} + 1 > 2k + j - i$. Assume that $i > h$ and let $\psi' = (D', \mu', S', i)$ be the most recent predecessor of ψ of level i. Then $(i, j-1) \in D'$ and $g' = j$. Since $\mu'_i = \mu_i$ and $\mu'_j \geq \mu_j$, $\mathrm{W}(i,j)$ holds for ψ'. But then ψ' meets **(iv)** and is not the most recent predecessor, a contradiction. □

Lemma 4.8. *Assume that $j > m$. If $h = 0$ or $(h,j) \in D$, then $\mu_j \leq \mu_{j-1}$.*

Proof. Assume that $\mu_j > \mu_{j-1}$. Then Lemmas 4.6 and 4.7(b) imply that $\mu_j = \lambda_{j-1} = \mu_{j-1} + 1$, and $D \smallsetminus \mathcal{C}$ contains a unique pair $(i, j-1)$ in column $j-1$, with $i \geq h$. Let $\psi' = (D', \mu', S', i)$ be the most recent predecessor of ψ of level

i for which $(i,j) \notin D'$. The assumptions of the lemma then imply that $\psi' \neq \psi$. Lemma 4.5 shows that $(i, j-1)$ was added to D by **(i)** or **(iii)**, so ψ' satisfies $\mathrm{W}(i, j-1)$. Since $\mu'_j \geq \mu_j > \mu_{j-1} = \mu'_{j-1}$, ψ' also satisfies $\mathrm{W}(i,j)$, so ψ' meets **(iii)** or **(iv)**. The choice of ψ' implies that (i,j) must be the outer corner added to D', so in fact ψ' meets **(iii)**. Now the statement of rule **(iii)** implies that $(i,j) \in S$, so $\mu'_j > \mu_j > \mu'_{j-1}$. This contradicts Corollary 4.1. □

Lemma 4.9. *Assume $h < m$. Then $\mu_j \leq \lambda_{j-1}$ and $\mu_j < \mu_{j-1}$ for $h < j \leq m$.*

Proof. If the statement is false, then choose $i > h$ minimal such that $\mu_i > \lambda_{i-1}$ or $\mu_i \geq \mu_{i-1}$, and let $\psi' = (D', \mu', S', i)$ be the most recent predecessor of ψ of level i. Since $\mu'_i > \lambda_{i-1}$ or $\mu'_i \geq \mu'_{i-1}$ and condition X fails for ψ', we have $(i,i) \notin D'$, so $i = m$ and $\mu'_m \geq \lambda_{m-1} > k$. But then $\mu'_t + \mu'_m \geq \lambda_t + \mu'_m > (k + m - t) + k$ for $1 \leq t < m$, so ψ' meets **(i)**, a contradiction. □

Lemma 4.10. *Assume that $(h,h) \in D$, $\mu_h = \lambda_{h-1}$, and $[h, \lambda_{h-1}] \in R$. Then ψ satisfies condition* X *and does not survive the algorithm.*

Proof. Since $[h, \lambda_{h-1}] \in R$ and $r(h + \lambda_{h-1}) = g$, we have $\mu_g \leq 2k + g - h - \lambda_{h-1}$, hence $\mathrm{W}(h,g)$ fails for ψ. Lemma 4.9 shows that $\mu_h > \mu_{h+1} > \cdots > \mu_m$, so $\mathrm{W}(d,g)$ fails for $h \leq d \leq m$. If $(h,g) \notin D$ then condition X holds because $(h,f) = (h,g) \notin S$. Otherwise choose $i \geq h$ maximal such that $(i,g) \in D$, and let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ for which $(i,g) \notin D'$. Then ψ' meets **(iv)** and ψ is a successor of the child $\psi'' = (D, \mu, S, h')$ of ψ'. If $h' > h$, then Lemma 4.4 implies that X fails and $\mathrm{W}(h', g)$ holds for ψ'. Since $\mathrm{W}(h', g)$ fails for ψ'', it follows that $i < h'$ and $\mu_{h'} + \mu_g = 2k + g - h'$. We also have $\lambda_{h'-1} + \mu_g \leq \lambda_{h-1} + \mu_g + h - h' \leq 2k + g - h'$, so $\mu_{h'} \geq \lambda_{h'-1}$, and using Lemma 4.3 we get $(h', f'') = (h', g) \notin S$. But then ψ'' satisfies condition X, a contradiction. We therefore have $h' = h$, $\mathrm{W}(h,g)$ fails for ψ', X holds for ψ', and the result follows from Lemma 4.4. □

In our applications of Lemma 4.10 we only need the fact that ψ does not survive the algorithm, so it is enough to know that a predecessor of ψ meets **(iv)**. The last 6 lines of the above proof could therefore be omitted.

4.3. In this section we will study a 4-tuples $\psi = (D, \mu, S, 0) \in \Psi_0$ with $\mu_{\ell+1} \geq 0$. For such a 4-tuple, Lemma 4.5 implies that each pair $(i,j) \in D \smallsetminus \mathcal{C}$ was added by **(i)** or **(iii)**. More precisely, the pair (i,j) was added by **(i)** if $i = j = m$ or $(i, j-1) \in \mathcal{C}$, and otherwise the pair was added by **(iii)**.

Proposition 4.1. *Suppose that $\psi = (D, \mu, S, 0)$ and $\mu_{\ell+1} \geq 0$. Then μ is a k-strict partition with $|\mu| = |\lambda| + p$, satisfying $\lambda_j - 1 \leq \mu_j \leq \lambda_{j-1}$ for every $j \geq 1$, and $\lambda_j \leq \mu_j$ when $\lambda_j > k$. Furthermore, we have $D = \mathcal{C}_{\ell+1}(\mu)$.*

Proof. By Lemma 4.9 we have $\mu_j \leq \lambda_{j-1}$ and $\mu_j < \mu_{j-1}$ for $1 \leq j \leq m$, and Lemmas 4.6 and 4.8 show that $\mu_j \leq \min(\lambda_{j-1}, \mu_{j-1})$ for $j > m$. We deduce that μ is a k-strict partition. Lemma 4.7(b) implies that $\lambda_j - 1 \leq \mu_j$ for every j. Clearly $\lambda_j \leq \mu_j$ when $\lambda_j > k$, and $|\mu| = |\lambda| + p$.

It remains to show that the set $\mathcal{C}_{\ell+1}(\mu)$ is equal to D. If $D \not\subset \mathcal{C}_{\ell+1}(\mu)$, then since $\mathcal{C}_{\ell+1}(\mu)$ and D are both valid sets of pairs, we can find an inner corner $(i,j) \in D \smallsetminus \mathcal{C}_{\ell+1}(\mu)$ such that $(i+1, j) \notin D$ and $(i, j+1) \notin D$. Since $(i,j) \notin \mathcal{C}$ by Lemma 2.1, the pair (i,j) was added by **(i)** or **(iii)**, and $\mathrm{W}(i,j)$ holds since μ_i and μ_j did not change since this event.

On the other hand, if $\mathcal{C}_{\ell+1}(\mu) \not\subset D$, then we can find an outer corner (i,j) of D such that $(i,j) \in \mathcal{C}_{\ell+1}(\mu)$. If $(i,j) = (m,m)$ or if $(i,j-1) \in \mathcal{C}$, then the most recent predecessor of ψ of level j meets **(i)**, and otherwise we deduce from Lemma 2.1 that the most recent predecessor of ψ of level i meets **(iii)**. This contradiction finishes the proof. □

Lemma 4.11. *Assume that $\psi = (D,\mu,S,0)$ and j are such that $\mu_j = \lambda_j - 1 \geq 0$, and let (i,j) be the unique pair in column j of $D \smallsetminus \mathcal{C}$. Then the removed box $[j,\lambda_j]$ and the above box $[j-1,\lambda_j]$ are k-related to the boxes $[i,c]$ and $[i,c-1]$, respectively, where $c = 2k+2+j-i-\lambda_j$, and these latter boxes belong to R.*

Proof. Let $\psi' = (D',\mu',S',h')$ be the most recent predecessor of ψ for which $(i,j) \notin D'$. Since $\mu'_j = \lambda_j$ and ψ' satisfies W(i,j), we obtain $\mu_i \geq \mu'_i + 1 \geq c$, and since $(i,j) \notin \mathcal{C}$ we similarly have $\lambda_i \leq c-2$. The boxes $[i,c]$ and $[i,c-1]$ belong to R because $\mu_{j+1} < \lambda_j$ and $\mu_j < \lambda_j$ (see §3.2). □

Lemma 4.12. *Assume that $\psi = (D,\mu,S,0)$, $\mu_{\ell+1} \geq 0$, and let $i \leq m$ be any integer such that $\mu_i = \lambda_{i-1}$. Then $[i,\lambda_{i-1}] \notin R$ and $(i, f_i(\mu)) \in S$.*

Proof. Let $\psi' = (D',\mu',S',i)$ be the most recent predecessor of ψ of level i. Then $\mu'_i = \mu_i$ and $(i,i) \in D'$; if $i = m$ this follows because $\mu'_m = \lambda_{m-1} > k$. Lemma 4.10 implies that $[i,\lambda_{i-1}] \notin R'$. Since all pairs $(c,d) \in D \smallsetminus D'$ were added by **(iii)** and satisfy $d > g'$, it follows that $\mu'_j = \mu_j$ for $m \leq j \leq g'$, so Lemma 4.2 shows that $[i,\lambda_{i-1}] \notin R$ and $f' = f_i(\mu)$. Finally, we must have $(i,f') \in S' \subset S$ since ψ' does not satisfy condition X. □

Proposition 4.2. *If $\psi = (D,\mu,S,0)$ and $\mu_{\ell+1} \geq 0$, then we have $\lambda \to \mu$.*

Proof. By Proposition 4.1, it suffices to check that conditions (1) and (2) of §2.2 are true. Condition (1) follows from Lemma 4.12 since $[i,\lambda_{i-1}] \notin R$ for each i. Suppose that $\mu_j + 1 = \lambda_j = d$ for $j_1 \leq j \leq j_2$. According to Lemma 4.11, each removed box $[j,d]$ for $j_1 \leq j \leq j_2$ is k-related to some box $[i_j,c_j] \in \mu \smallsetminus \lambda$, and the box $[i_j,c_j-1]$ is also in $\mu \smallsetminus \lambda$. Condition (1) implies that each box $[j,d]$ is k-related to at most one box of $\mu \smallsetminus \lambda$. It follows that if $j < j_2$, then $[i_j,c_j] = [i_{j+1},c_{j+1}-1]$, so all the boxes $[i_j,c_j]$ lie in the same row of $\mu \smallsetminus \lambda$. Condition (2) follows from this since we also know that the box $[j_1-1,d]$ is k-related to $[i_{j_1},c_{j_1}-1]$. □

4.4. Propositions 4.1 and 4.2 tell us that if $\psi = (D,\mu,S,0)$ is any 4-tuple in Ψ_0 with $\mu_{\ell+1} \geq 0$, then μ is a k-strict partition with $\lambda \to \mu$, $D = \mathcal{C}_{\ell+1}(\mu)$ is uniquely determined by μ, and $\mathrm{ev}(\psi) = T(\mathcal{C}(\mu),\mu)$ is a term appearing in the Pieri rule (14). To account for the multiplicities, we give an explicit construction of the possible sets S in these 4-tuples. In this section we fix an arbitrary k-strict partition μ such that $\lambda \to \mu$ and $|\mu| = |\lambda| + p$.

A *component* means an (edge or vertex) connected component of the set $\mathbb{A}$ of §2.2. We say that a box B of $\mathbb{A}$ is *distinguished* if the box directly to the left of B does not lie in $\mathbb{A}$. We say that B is *optional* if it is the rightmost distinguished box in its component. Notice that $\mathfrak{N}(\lambda,\mu)$ is equal to the number of optional distinguished boxes in $\mathbb{A}$. To each distinguished box $B = [i,c]$ we associate the pair $(i,j) = (i, r(i+c))$. The inequality $\lambda_{i-1} > 2k+i-(i+c)$ implies that $i \leq j$, so $(i,j) \in \Delta$. Let E (respectively F) be the set of pairs associated to optional (respectively non-optional) distinguished boxes. We furthermore let G be the set of all pairs $(i,j) \in \Delta$ for which some box in row i of $\mu \smallsetminus \lambda$ is k-related to a box in row j of $\lambda \smallsetminus \mu$.

Lemma 4.13. (a) *We have* $E \cup F \cup G \subset \mathcal{C}_{\ell+1}(\mu) \cap \partial\mathcal{C}$.
(b) *Each pair in* $E \cup F$ *is associated to exactly one distinguished box of* $\mathbb{A}$.
(c) *The sets* E, F, *and* G *are pairwise disjoint.*
(d) *If* $(i,j) \in F$, *then* $j = f_i(\mu)$.
(e) *If* $(i,j) \in E \cup F \cup G$, $i < j$, *and* $(i, j-1) \notin \mathcal{C}$, *then* $\mu_j < \lambda_{j-1}$.

Proof. Let $(i,j) \in G$. Then $\mu_j = \lambda_j - 1$ and the boxes $[j, \lambda_j]$ and $[j-1, \lambda_j]$ are k-related to $[i,d]$ and $[i, d-1]$, where $d = 2k+2+j-i-\lambda_j$. We also have $\lambda_i + 1 < d \leq \mu_i$. Therefore $\lambda_i + \lambda_j < d + \lambda_j - 1 = 2k+1+j-i$ and $\mu_i + \mu_j \geq d + \mu_j = 2k+1+j-i$, so $(i,j) \in \mathcal{C}_{\ell+1}(\mu) \smallsetminus \mathcal{C}$. Assume that (i,j) is associated to a distinguished box $[i,c] \in \mathbb{A}$. Since $r(i+c) = j < j+1 = r(i+d)$, we must have $c < d$, hence $\mu_j = 2k+1+j-i-d \leq 2k+j-i-c$. But then $[i,c] \in R(\mu)$, a contradiction. It follows that $G \cap (E \cup F) = \emptyset$.

Now let $[i,c] \in \mathbb{A}$ be distinguished and set $j = r(i+c)$. Then $j \geq r(i+\lambda_i+1)$, so (16) shows that $(i,j) \notin \mathcal{C}$. Since $[i,c] \notin R(\mu)$ we get $\mu_j > 2k+j-i-c \geq 2k+j-i-\mu_i$, hence $(i,j) \in \mathcal{C}_{\ell+1}(\mu)$. Using Lemma 2.1, this establishes (a). If $[i,c'] \in R(\mu)$ is any box with $c' > c$, then we must have $j < r(i+c')$, since otherwise $\mu_j \leq 2k+j-i-c' < 2k+j-i-c$. This proves (b) and finishes the proof of (c). If $(i,j) \in F$, then $\mu_i = \lambda_{i-1}$, $c = e_i(\mu)$, and $j = f_i(\mu)$. This establishes part (d).

In the situation of (e), notice that if $\mu_j = \lambda_{j-1}$, then $(i,j) \in E \cup F$ is associated to a distinguished box $[i,c] \in \mathbb{A}$. We must have $c > k+1$, since otherwise $\lambda_i \leq k$, $i = m$, and $j = r(m+k+1) = i$. Since $[i,c] \notin R(\mu)$ and $(i,j-1) \notin \mathcal{C}$, we also have $c > 2k+j-i-\mu_j = 2k+j-i-\lambda_{j-1} \geq \lambda_i + 1$. It follows that $[i, c-1] \in R(\mu)$. Set $j' = r(i+c-1)$. Then $j' \leq j$ and $\mu_{j'} - j' \leq 2k - i - c + 1 \leq \mu_j - j$, which shows that $j' = j$ and $\mu_j = 2k+j-i-c+1 < \lambda_{j-1}$. This contradiction proves (e). ☐

To every subset E' of E we define the set of pairs $S(E') := E' \cup F \cup G$. This is a disjoint union, and there are exactly $2^{\mathfrak{N}(\lambda,\mu)}$ sets of this form. The following proposition therefore completes the proof of Claim 1.

Proposition 4.3. *Let* $S \subset \Delta$ *be any subset. Then* $(\mathcal{C}_{\ell+1}(\mu), \mu, S, 0) \in \Psi_0$ *if and only if* $S = S(E')$ *for some subset* $E' \subset E$.

Proof. We first assume that $\psi = (\mathcal{C}_{\ell+1}(\mu), \mu, S, 0) \in \Psi_0$. Lemmas 4.12 and 4.13(d) then imply that $F \subset S$. We next show that $G = \{(i,j) \in S \mid \mu_j < \lambda_j\}$. If $\mu_j < \lambda_j$ then S contains a unique pair (i,j) in column j. Lemma 4.11 shows that $[j, \lambda_j]$ is k-related to a box in row i of $\mu \smallsetminus \lambda$, and condition (1) implies that no other box in $\mu \smallsetminus \lambda$ is k-related to $[j, \lambda_j]$. It follows that (i,j) is also the unique pair of G in column j.

Let $(i,j) \in S \smallsetminus G$. We will show that (i,j) is the pair associated to a distinguished box of $\mathbb{A}$. If $i = j = m$, then $\lambda_m \leq k < \mu_m$ and (m,m) is associated to the distinguished box $[m, k+1] \in \mathbb{A}$. We can therefore assume that $i < j$, hence $\mu_i > \lambda_i$. Since $(i,j) \notin G$ we also have $\lambda_j \leq \mu_j$. If $\lambda_i + \mu_j \geq 2k+j-i$, then the inequality $\lambda_{j-1} \geq \mu_j > 2k+j-i-\lambda_i-1$ implies that $j \leq r(i+\lambda_i+1)$. Since $(i,j) \notin \mathcal{C}$, it follows from (16) that $j = r(i+\lambda_i+1)$. We deduce that $[i, \lambda_i+1] \in \mathbb{A}$ is a distinguished box and (i,j) is the associated pair.

Otherwise we have $\lambda_i + \mu_j < 2k+j-i$. In this case we set $c = 2k+j-i-\mu_j$. Since $(i,j) \in \mathcal{C}_{\ell+1}(\mu)$ we have $\lambda_i < c < \mu_i$. We also have $c > k$; if $i = m$ this follows because $\mu_j \leq \lambda_m \leq k$. We claim that $\mu_j < \lambda_{j-1}$. If $(i, j-1) \in \mathcal{C}$, then this follows because $\mu_j < 2k + j - i - \lambda_i \leq \lambda_{j-1}$, so assume that $(i, j-1) \notin \mathcal{C}$. Then we must have $j > m$, and (i, j) was added to S in Phase 2 of the algorithm. By Lemma 4.6 the first predecessor (D', μ', S', i) of ψ of level i satisfies that $\mu'_j \leq \lambda_{j-1}$. Since $(i, j) \notin S'$, this implies that $\mu_j < \lambda_{j-1}$, as claimed.

The inequality $\lambda_{j-1} > \mu_j = 2k + j - i - c$ implies that $r(i + c) \geq j$, and since $\lambda_j \leq \mu_j$ we also have $r(i + c + 1) \leq j$. We deduce that $r(i + c) = r(i + c + 1) = j$, $[i, c] \in R(\mu)$, and $[i, c + 1] \in \mathbb{A}$. This shows that $[i, c + 1]$ is distinguished and (i, j) is the associated pair. We conclude that the set $E' := S \smallsetminus (F \cup G)$ is a subset of E, hence $S = S(E')$ has the required form.

Now let $E' \subset E$ be an arbitrary subset and set $S = S(E')$. We must show that $(\mathcal{C}_{\ell+1}(\mu), \mu, S, 0) \in \Psi_0$. Set $\nu = \prod_{(i,j)\in S} L_{ij}\, \mu$. The definition of S ensures that $\nu \geq \lambda$, so $\nu \in \mathcal{N}(\lambda, p)$. We now construct a path $\mathcal{P}$ in the substitution forest by applying the substitution rule of §3.3 repeatedly to the initial 4-tuple $(\mathcal{C}, \nu, \emptyset, \ell+1)$. Whenever the substitution rule assigns two children to a 4-tuple ψ' of $\mathcal{P}$, we use the set S to determine which child is to follow ψ' on the path. More precisely, if $\psi' = (D', \mu', S', h')$ meets **(i)** or **(iii)** and has two children, and if (i, j) is the outer corner being added to D', then we choose the child $\psi'' = (D'', \mu'', S'', h')$ for which $S'' \smallsetminus S' = S \cap \{(i, j)\}$. We will show that $\mathcal{P}$ terminates in the 4-tuple $(\mathcal{C}_{\ell+1}(\mu), \mu, S, 0)$.

For $h \geq 0$ we set $D_h = \mathcal{C} \cup \{(i, j) \in \mathcal{C}_{\ell+1}(\mu) \mid i > h \text{ or } (j > h \text{ and } (i, j-1) \in \mathcal{C})\}$. Lemma 2.1 implies that this is a valid set of pairs. We will say that the 4-tuple $\psi' = (D', \mu', S', h')$ is *good* if it satisfies $D_{h'} \subset D' \subset D_{h'-1}$ and $S' = S \cap D'$. It is enough to show that if ψ' is any good 4-tuple on $\mathcal{P}$ with $h' > 0$, then ψ' has a good child that also belongs to $\mathcal{P}$.

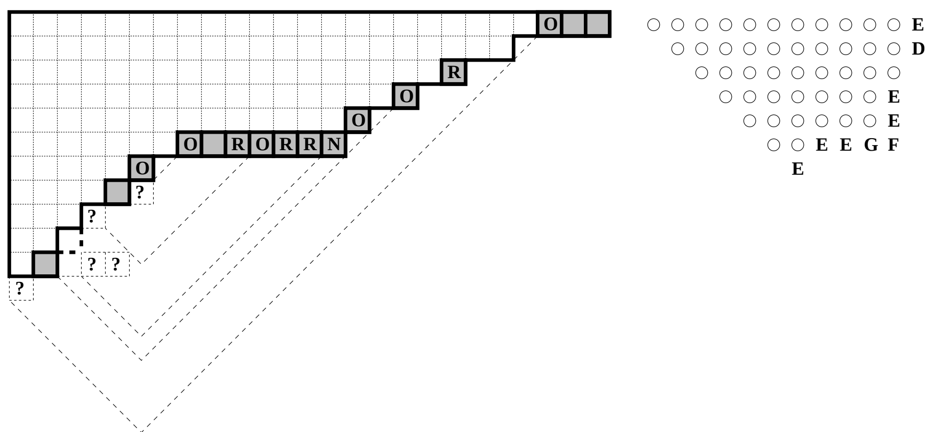
Let ψ' be a good 4-tuple of $\mathcal{P}$. We then have $\mu = \prod_{(i,j)\in S\smallsetminus D'} R_{ij}\, \mu'$. We first show that if ψ' meets **(i)** or **(iii)**, and (i, j) is the pair being added to D', then (i, j) is also in $D_{h'-1}$. If $\mu_i + \mu_j = \mu'_i + \mu'_j$ then this is true because ψ' satisfies $\mathrm{W}(i, j)$, and otherwise $S \smallsetminus D'$ must contain at least one pair in row i or column j, which implies that $(i, j) \in D_{h'-1}$ by Lemma 4.13(a). On the other hand, assume that $D' \subsetneq D_{h'-1}$. Then $D_{h'-1} \smallsetminus D_{h'}$ contains an outer corner (i, j) of D'. If we choose $c \geq j$ maximal such that $(i, c) \in \mathcal{C}_{\ell+1}(\mu)$, then the inequalities $\mu'_i + \mu'_j \geq \mu_i + \mu_j - (c - j) \geq \mu_i + \mu_c - c + j > 2k + j - i$ show that ψ' satisfies $\mathrm{W}(i, j)$. If $h' = j$ then ψ' meets **(i)**, and otherwise we have $h' = i < j$ and ψ' meets **(iii)**. Notice also that if ψ' meets **(iii)** and $\mu'_j > \mu'_{j-1}$, then we must have $(h', j) \in S$ since μ is a partition. These observations show that ψ' meets **(i)** or **(iii)** if and only if $D' \subsetneq D_{h'-1}$, and in this case ψ' is succeeded on $\mathcal{P}$ by a good child.

Now consider a good 4-tuple ψ' of $\mathcal{P}$ such that $D' = D_{h'-1}$. It remains to show that the substitution rule simply decreases the level of ψ', i.e. ψ' does not meet **(ii)**, **(iv)**, or **(v)**. Assume that ψ' meets **(ii)** and choose $i \geq 1$ minimal such that $(i, h') \notin D'$. Then $\mu'_{h'} > \lambda_{h'-1}$ and (i, h') is not an outer corner of D'. We have $i < h'$ and $(i, h'-1) \notin \mathcal{C}$. Using Lemma 2.1 we deduce that $(i, h') \in S$ and $\mu_{h'} = \lambda_{h'-1}$; however this contradicts Lemma 4.13(e).

If ψ' satisfies condition X, then $h' \leq m$ and $\mu'_{h'} = \mu_{h'} = \lambda_{h'-1}$. It follows that $\mathbb{A}$ contains a non-optional distinguished box in row h', and Lemma 4.13(d) implies that $(h', f_{h'}(\mu)) \in F$ is the associated pair. But Lemma 4.2 shows that $f' = f_{h'}(\mu)$, so $(h', f') \in S \cap D_{h'-1} = S'$. We conclude that condition X fails for ψ'. In particular, ψ' does not meet **(v)**. Finally, if ψ' meets **(iv)**, then $(h', g') \notin \mathcal{C}_{\ell+1}(\mu)$, and since

$\mu'_{h'} = \mu_{h'}$ and $\mu'_{g'} = \mu_{g'}$, we deduce that $\mathrm{W}(h', g')$ fails for ψ'. This contradiction finishes the proof that the level of ψ' is decreased. □

Example 4.1. Consider the partitions $\lambda = (22, 21, 18, 16, 14, 7, 5, 4, 3, 3, 1)$ and $\mu = (25, 21, 19, 17, 15, 14, 6, 5, 3, 2, 2)$, and set $k = 5$. Then $\lambda \to \mu$. The diagrams of λ and μ are displayed in the left half of the figure below; boxes from $R(\mu)$ are labeled with **R** and distinguished boxes are labeled with **O** for optional and **N** for non-optional. The right half of the figure shows the pairs in $\mathcal{C}(\lambda)$ and $\mathcal{C}_{\ell+1}(\mu)$, where $\ell = 11$. The pairs from E, F, and G are labeled accordingly, and the one additional pair from $\mathcal{C}_{\ell+1}(\mu) \smallsetminus \mathcal{C}(\lambda)$ is labeled **D**. The skew dotted lines on the left half help to identify the pairs in E and F associated to the distinguished boxes. The compositions ν for which some 4-tuple $(\mathcal{C}_{\ell+1}(\mu), \mu, S, 0)$ originates from $(\mathcal{C}, \nu, \emptyset, \ell{+}1)$ may or may not include the boxes labelled with question marks, which can be traded for boxes from the rows of corresponding optional distinguished boxes. There are 2^5 such compositions ν, and for each of them there are two sets S, one of which contains the diagonal pair $(7, 7) \in E$.



4.5. In this section, we define the involution $\iota : \Psi_1 \to \Psi_1$ and prove Claim 2. Suppose that a 4-tuple $\psi = (D, \mu, S, h) \in \Psi_1$ is such that $(h, h) \notin D$, and thus met condition **(ii)** in the course of the algorithm. Define μ' by setting $\mu'_{h-1} = \mu_h - 1$, $\mu'_h = \mu_{h-1} + 1$, and $\mu'_r = \mu_r$ for $r \notin \{h-1, h\}$. Let

$$\iota(D, \mu, S, h) = (D, \mu', S, h);$$

then $\mathrm{ev}(\psi) + \mathrm{ev}(\iota(\psi)) = 0$, using Lemma 1. If ν and ν' are the initial compositions that gave rise to (D, μ, S, h) and (D, μ', S, h), respectively, then note that $\mu'_h > \lambda_{h-1}$ and $\nu_j = \nu'_j$ for $j \notin \{h-1, h\}$. Therefore it is easy to verify that $\nu' \in \mathcal{N}(\lambda, p)$ and that $(\mathcal{C}, \nu', \emptyset, \ell+1)$ goes through the same branching history to reach (D, μ', S, h) as $(\mathcal{C}, \nu, \emptyset, \ell+1)$ went through to reach (D, μ, S, h). We deduce that (D, μ', S, h) also meets condition **(ii)**, and hence lies in Ψ_1; clearly $\iota(D, \mu', S, h) = (D, \mu, S, h)$.

Suppose next that a term $\psi = (D, \mu, S, h) \in \Psi_1$ satisfies $(h, h) \in D$, and thus met condition **(v)**. Since $(h, g) \in D$, the set D does not have an outer corner in row h, and Lemma 1.2 applies. If $\mu_h = \mu_{h-1}$, we let $\iota(D, \mu, S, h) = (D, \mu, S, h)$. Otherwise, let ϖ be the involution on Δ which exchanges $(h-1, g)$ with (h, f), and fixes all other pairs. Define μ' by switching parts μ_{h-1} and μ_h in the composition μ, set

$S' = \varpi(S)$, and let $\iota(D,\mu,S,h) = (D,\mu',S',h)$. Then we have $\mathrm{ev}(\psi)+\mathrm{ev}(\iota(\psi)) = 0$, for every $\psi \in \Psi_1$, by construction.

Lemma 4.2 implies that the same values of e, f, and g are assigned to the two terms $\psi = (D,\mu,S,h)$ and $\psi' = (D,\mu',S',h)$. It follows that $\iota(\psi') = \psi$. We claim that ψ' also satisfies condition X. Indeed, if $\mu_{h-1} = \lambda_{h-1} < \mu_h$, then we must have $(h-1,g) \notin S$, and hence X holds for ψ', since we have $\mu'_h = \lambda_{h-1}$ and $(h,f) \notin S'$. In all other cases, the claim is clear.

It remains to show that $\psi' \in \Psi_1$. We assume in the following that ψ meets **(v)**, and will prove that ψ' also meets **(v)**.

Lemma 4.14. *We have* $\lambda_{h-1} - \lambda_h \geq g - b + 1$.

Proof. The inequality is clear if $b = g$, as λ is k-strict and $h \leq m$. If $b < g$, then since $b = r(h+\lambda_h+1)$ and $g = r(h+\lambda_{h-1})$ we have $\lambda_b \leq 2k+b-h-\lambda_h$ and $\lambda_{g-1} > 2k+g-h-\lambda_{h-1}$, which implies that $\lambda_{h-1}-\lambda_h > g-\lambda_{g-1}-b+\lambda_b \geq g-b$. □

Lemma 4.15. *If $f > m$, then $\mu_f < \lambda_{f-1}$, or $\mu_f = \lambda_{f-1}$ and $f = b$.*

Proof. Since $f > m \geq h$, Lemma 4.6 implies that $\mu_f \leq \lambda_{f-1}$. Suppose that $\mu_f = \lambda_{f-1}$. If $(h,f-1) \in \mathcal{C}$, then $h < m$ and hence $f = b$. We may therefore assume that $(h,f-1) \notin \mathcal{C}$. If $[h,\lambda_{h-1}] \in R(\mu)$ then $f = g = r(h+\lambda_{h-1})$, and the inequalities

$$\mu_g \leq 2k+g-h-\lambda_{h-1} < \lambda_{g-1} = \mu_g$$

give a contradiction. We must have $e > k+1$, since otherwise $\lambda_h \leq k$, $h = m$, and $f = r(m+k+1) = m$. Since $[h,e] \notin R(\mu)$ and $(h,f-1) \notin \mathcal{C}$, we have $e > 2k+f-h-\mu_f = 2k+f-h-\lambda_{f-1} \geq \lambda_h+1$, and therefore $[h,e-1] \in R(\mu)$. Set $f_1 = r(h+e-1)$. Then $f_1 \leq f$ and $\mu_{f_1}-f_1 \leq 2k-h-e+1 \leq \mu_f-f$. Now Lemma 4.8 implies that $\mu_f \leq \mu_{f_1}$, therefore $f_1 = f$ and $\mu_f = 2k+f-h-e+1 < \lambda_{f-1}$. This contradiction completes the proof. □

Lemma 4.16. *If $f < g$ then $\mu_g = \lambda_{g-1}$, $(h,g) \notin S$, and ψ, ψ' both satisfy* $\mathrm{W}(h,g)$.

Proof. Since $f < g$, we must have $(h,g-1) \notin \mathcal{C}$ and $g > m$, so $\mu_g \leq \lambda_{g-1}$ by Lemma 4.6. Note that $c := 2k+1+g-h-\lambda_{h-1} \leq \lambda_{g-1}$, since $(h-1,g-1) \in \mathcal{C}$. If $\mu_g < c$, then $[h,\lambda_{h-1}] \in R$, which contradicts $f < g$. Suppose next that $c \leq \mu_g < \lambda_{g-1}$, and let $x = 2k+g-h-\mu_g > k$. Observe that

$$\lambda_h + 1 \leq 2k+g-h-\lambda_{g-1} < x \leq 2k+g-h-c = \lambda_{h-1}-1.$$

As box $[g-1,\mu_g+1]$ is k-related to $[h,x]$, we have $[h,x] \in R$ and $e = x+1$, i.e., $\mu_g = 2k+1+g-h-e$, and therefore $f = r(h+e) = g$, a contradiction. We deduce that $\mu_g = \lambda_{g-1}$ and $(h,g) \notin S$. As ψ satisfies X, we have $\mu_h \geq \lambda_{h-1}$, therefore

$$\mu_h + \mu_g \geq \lambda_{h-1} + \lambda_{g-1} > 2k+g-h \tag{18}$$

and condition $\mathrm{W}(h,g)$ holds. The inequalities (18) are also true for ψ'. □

Lemma 4.17. *Suppose that $(h,g-1) \notin \mathcal{C}$ and define $q = \#\{i > h \,|\, (i,g-1) \in S\}$. Then $\mu'_h+\mu'_{g-1}+q \geq 2k+g-h$, and if $(h,g) \in S'$, we have $\mu'_h+\mu'_{g-1}+q > 2k+g-h$.*

Proof. Condition X for ψ' implies that $\mu'_h \geq \lambda_{h-1}$. Since $\mu'_{g-1} \geq \lambda_{g-1}-1-q$ and $(h-1,g-1) \in \mathcal{C}$, we deduce that $\mu'_h + \mu'_{g-1} + q \geq \lambda_{h-1} + \lambda_{g-1} - 1 \geq 2k+g-h$. We show next that equality implies $(h,g) \notin S'$, so suppose that $\mu'_h = \lambda_{h-1}$ and $\mu'_{g-1} = \lambda_{g-1}-1-q$. In this case $g-1 > m$, and Lemma 4.8 implies that

$\mu_g \leq \mu_{g-1} = \mu'_{g-1} < \lambda_{g-1}$. By Lemma 4.16 we have $f = g$, and, since ψ' satisfies X, it follows that $(h, g) \notin S'$. □

Define the retraced weight condition $\widetilde{\mathrm{W}}(r, g)$ on a valid 4-tuple $(\overline{D}, \overline{\mu}, \overline{S}, \overline{h})$ by the inequality

$$\widetilde{\mathrm{W}}(r, g) \ : \ \overline{\mu}_r + \overline{\mu}_g + N_r > 2k + g - r,$$

where $N_r = \#\{i > r \mid (i, g) \in \overline{S}\}$. For the 4-tuple ψ, let z be the least integer with $a \leq z \leq h - 1$ such that $\widetilde{\mathrm{W}}(z, g)$ fails; if no such integer exists, then set $z = h$. If $z < h$, it follows that the pairs $(a, g), \ldots, (z - 1, g)$ were added to D in Phase 1 of the algorithm, and the pairs $(z, g), \ldots, (h, g)$ were added to D in Phase 2. We deduce by arguing as in Lemma 4.6 that the inequality

$$\mu_g + N_{z-1} \leq \lambda_{g-1} \tag{19}$$

holds. Define the integers N'_r and z' associated to the 4-tuple ψ' in the same way that N_r and z are associated to ψ.

Proposition 4.4. *Assume that $z \neq z'$.*

(a) *If $f = g$, then $\{z, z'\} = \{h - 1, h\}$, and either ψ or ψ' satisfies $\widetilde{\mathrm{W}}(h, g)$.*

(b) *If $f < g$, then $\max(z, z') = h$, and ψ, ψ' both satisfy $\mathrm{W}(h, g)$.*

Proof. If $f = g$, then from the definition of S' we obtain that $N_r = N'_r$ for any $r < h-1$. We deduce that $z < h-1$ if and only if $z' < h-1$, in which case we have $z = z'$. Since $z \neq z'$, it follows that $\{z, z'\} = \{h - 1, h\}$. If $z = h$, then $\widetilde{\mathrm{W}}(h - 1, g)$ holds for ψ, and therefore $\widetilde{\mathrm{W}}(h, g)$ holds for ψ'. This proves part (a).

If $f < g$, we know by Lemma 4.16 that ψ and ψ' both satisfy $\mathrm{W}(h, g)$ and $(h, g) \notin S \cup S'$. Assume that $(h - 1, g) \in S$. Since $\mu_g = \lambda_{g-1}$, Lemma 4.6 for the term ψ shows that the pair $(h - 1, g)$ was added to D in Phase 1 of the algorithm, and hence $z = h$. A similar observation applies to the term ψ'. On the other hand, if $(h - 1, g) \notin S \cup S'$, then reasoning as in the proof of (a), we see that $\{z, z'\} = \{h - 1, h\}$. □

Proposition 4.5. *If $(h + 1, g) \in D$, then both ψ and ψ' satisfy condition $\widetilde{\mathrm{W}}(i, g)$ for $a \leq i \leq h + 1$. In particular, we have $f = g$ and $z = z' = h$.*

Proof. Clearly $f = g = b$, and Lemma 4.5 implies that $(h + 1, g)$ was added to D in Phase 1 of the algorithm for ψ. Therefore ψ satisfies $\widetilde{\mathrm{W}}(i, g)$ whenever $a \leq i \leq h + 1$, and $z = h$. It is clear from the definitions that $\widetilde{\mathrm{W}}(i, g)$ holds for ψ' when $a \leq i \leq h + 1$ and $i \notin \{h - 1, h\}$. It is also easy to see that condition $\widetilde{\mathrm{W}}(h - 1, g)$ for ψ implies that $\widetilde{\mathrm{W}}(h, g)$ holds for ψ'. Notice that $\mu_h > \lambda_h$, since ψ meets **(v)** and therefore satisfies X. Moreover, $\widetilde{\mathrm{W}}(g, h + 1)$ holds for ψ, and hence

$$\mu_h + \mu_g + N_{h+1} > \lambda_h + \mu_g + N_{h+1} \geq \mu_{h+1} + \mu_g + N_{h+1} \geq 2k + g - h. \tag{20}$$

We claim that $\mu_h + \mu_g + N_h > 2k + 1 + g - h$. This follows immediately from (20) if $\mu_{h+1} < \lambda_h$. If $\mu_{h+1} = \lambda_h$, then we must have $(h + 1, g) \in S$, for otherwise the parent of ψ would have met **(v)**. We conclude that $N_h = N_{h+1} + 1$, proving the claim. Since $N'_h = N_h$ and $\mu'_g = \mu_g$, we obtain $\mu'_{h-1} + \mu'_g + N'_h > 2k + g - (h - 1)$, which is condition $\widetilde{\mathrm{W}}(h - 1, g)$ for ψ'. □

We will obtain a sequence of valid 4-tuple predecessors $(\overline{D}, \overline{\mu}, \overline{S}, h)$ of ψ' by successively removing pairs (i, g) for $i \leq h$ from the set $D \cap \partial\mathcal{C}$, and applying corresponding lowering operators to μ', as dictated by the set S'. This backtracking continues until weight considerations along column g force the sequence of predecessors to proceed by removing pairs along row h, if $(h, g-1) \notin \mathcal{C}$, or by increasing h to $h+1$, if $(h, g-1) \in \mathcal{C}$. The precise point when this happens is specified by the value of z'. If $z = z'$, then the backtracking sequence for the term ψ' is essentially the same as that for ψ. The various possibilities when $z \neq z'$ are explained in Proposition 4.4; in this case either ψ or ψ' is such that each of the pairs $(a, g), \ldots, (h-1, g)$ will be added to D in Phase 1.

We claim that all predecessors $\overline{\psi} = (\overline{D}, \overline{\mu}, \overline{S}, h)$ of ψ' with $(h, g-1) \in \overline{D}$ satisfy either W(h, g) or X. If condition W(h, g) is true for ψ' then it will also hold for all predecessors $\overline{\psi}$, so assume that W(h, g) is false for ψ'. We have $f = g$ by Lemma 4.16, and since W(h, g) fails we deduce that $[h, \lambda_{h-1}] \in R$. Let $\overline{f}$ and $\overline{R}$ be the values of f and R as computed for some predecessor $\overline{\psi}$ of ψ'. The only way that X can fail for $\overline{\psi}$ is if $\overline{\mu}_h = \lambda_{h-1}$ and $(h, \overline{f}) \in \overline{S}$. Clearly $\overline{f} < g$, therefore $[h, \lambda_{h-1}] \notin \overline{R}$, and we conclude that condition W(h, g) must hold for $\overline{\psi}$.

For the remainder of the proof of Claim 2, we distinguish two cases.

Case 1. Assume that $(h, g-1) \notin \mathcal{C}$. Then the backtracking sequence for ψ' begins by removing the pair (h, g) from D. We have seen that if $\mu_{h-1} = \lambda_{h-1} < \mu_h$, then ψ' satisfies $\mu'_h = \lambda_{h-1}$ and $(h, f) \notin S'$. On the other hand, if $\mu_h = \lambda_{h-1} < \mu_{h-1}$, we have $(h, f) \notin S$ and therefore $\mu'_{h-1} = \lambda_{h-1} < \mu'_h$ and $(h-1, g) \notin S'$. If $z' < h$, the sequence continues by successively removing the pairs $(h-1, g), \ldots, (z', g)$ from D. In this way, we arrive at a 4-tuple $(\overline{D}, \overline{\mu}, \overline{S}, h)$.

At this juncture, Lemma 4.17 ensures that $\overline{\mu}_h + \overline{\mu}_{g-1}$ is large enough so that the sequence can be traced back further by removing pairs along row h of $\overline{D}$ until at most one pair remains in row h of $\overline{D} \cap \partial\mathcal{C}$. To see this, we may assume that $g-1 > m$. Consider a predecessor $(\widetilde{D}, \widetilde{\mu}, \widetilde{S}, h)$ of $(\overline{D}, \overline{\mu}, \overline{S}, h)$ and $j \leq g-1$ maximal such that $(h, j) \in \widetilde{D}$. Then we have

$$\widetilde{\mu}_h + \widetilde{\mu}_j \geq \overline{\mu}_h + \overline{\mu}_j - (g-1-j) \geq \overline{\mu}_h + \overline{\mu}_{g-1} + 1 + j - g > 2k + j - h,$$

where the last two inequalities follow from Lemmas 4.8 and 4.17, respectively.

From the above point onwards, the backtracking process for ψ' continues exactly as it did for the 4-tuple ψ, until we reach the initial term $(\mathcal{C}, \nu', \emptyset, \ell+1)$ with $\nu' = \prod L_{ij}\, \mu'$, where the product is over all pairs $(i, j) \in S'$. Indeed, the sets S and S' only differ (potentially) in the pairs $(h-1, g)$ and (h, f), and correspondingly the parts of μ, μ' and their predecessors can only differ in rows $h-1$, h, f, and g. To verify that $\nu' \in \mathcal{N}(\lambda, p)$, we use this observation and Lemma 4.14, which implies that $\#\{j \mid (h, j) \in S'\} \leq \mu'_h - \lambda_h$. It is also easy to see from this that no predecessor of ψ' meets **(v)**. Finally, Lemma 4.15 and (19) ensure that the 4-tuples in the substitution path from $(\mathcal{C}, \nu', \emptyset, \ell+1)$ to ψ' do not meet **(ii)**.

Case 2. Assume that $(h, g-1) \in \mathcal{C}$. If $(h+1, g) \notin D$, then the backtracking sequence for ψ' is similar to that in Case 1. However, (h, g) is the only pair in row h of $D \cap \partial\mathcal{C}$, and if $z' = h < m$ and W(h, g) holds, then the parent of ψ' is $(D, \mu', S', h+1)$. If $(h+1, g) \in D$, then Proposition 4.5 shows that the pairs $(a, g), \ldots, (h, g)$ were added to D in Phase 1 of the algorithm for both ψ and ψ'.

In particular, the parent of ψ' is $(D, \mu', S', h+1)$, and the 4-tuples ψ, ψ' proceed through the algorithm in the same fashion.

This completes the proof of Claim 2, and of Theorem 1.

Example 4.2. The most subtle ingredient of the Substitution Rule is the condition $\mathrm{W}(h,g)$ in **(iv)**. This example shows that if we omit it from **(iv)**, then our cancellation scheme fails, and also illustrates the discussion in the paragraph just before Case 1 above.

Let $\lambda = (4,3,1)$, $p=4$, $k=1$, and take $n \geq 5$. We have $\mathcal{C} = \{11,12,13,22,23\}$. Consider the following sequence of 4-tuples in the algorithm, stemming from the root $\psi_0 = (\mathcal{C}, 4341, \emptyset, 4)$.

$$\psi_0 \longrightarrow (\mathcal{C}, 4341, \emptyset, 3) \xrightarrow{\text{(i)}} (\mathcal{C}\cup\{33\}, 4341, 33, 3) \xrightarrow{\text{(iv)}} (\mathcal{C}\cup\{33,14\}, 534, \{33,14\}, 3)$$

$$\xrightarrow{\text{(iv)}} (\mathcal{C}\cup\{33,14,24\}, 534, \{33,14\}, 3) \xrightarrow{\text{(iv)}} (\mathcal{C}\cup\{33,14,24,34\}, 534, \{33,14\}, 3) = \psi.$$

The substitution rule STOPS at the leaf ψ, for which we have $[3,3] \in R$ and hence $f = g = 4$. Now $\psi' = (\mathcal{C}\cup\{33,14,24,34\}, 543, \{33,14\}, 3)$, which backtracks to the initial term $\psi'_0 = (\mathcal{C}, 4431, \emptyset, 4)$. However, the tree of the substitution forest with root ψ'_0 contains the initial path

$$\psi'_0 \longrightarrow (\mathcal{C}, 4431, \emptyset, 3) \xrightarrow{\text{(i)}} (\mathcal{C}\cup\{33\}, 4431, 33, 3)$$

Note that for the term $\overline{\psi} = (\mathcal{C}\cup\{33\}, 4431, 33, 3)$ we have $\overline{f} = 3$, hence condition X fails. If the condition $\mathrm{W}(h,g)$ is omitted from **(iv)**, we deduce that $\overline{\psi}$ does not meet any of **(i)**–**(v)**, and is REPLACED by $(\mathcal{C}\cup\{33\}, 4431, 33, 2)$. We conclude that ψ' does not appear in the substitution forest, and the cancellation scheme fails.

5. Theta Polynomials

5.1. In this section we develop the theory of theta polynomials systematically; the exposition is influenced by that in Macdonald's text [M, III.8]. Let $x = (x_1, x_2, \ldots)$ and let $\Lambda = \Lambda(x)$ be the ring of symmetric functions in x. Consider the generating functions

$$E(x\,;t) = \prod_{i=1}^{\infty}(1+x_it) = \sum_{r=0}^{\infty} e_r(x)t^r \quad \text{and} \quad H(x\,;t) = \prod_{i=1}^{\infty}\frac{1}{1-x_it} = \sum_{r=0}^{\infty} h_r(x)t^r$$

for the elementary and complete symmetric functions e_r and h_r, respectively. Fix an integer $k \geq 0$, let $y = (y_1, \ldots, y_k)$, and for each r define $\vartheta_r = \vartheta_r(x\,;y)$ by

$$\vartheta_r = \sum_{i\geq 0} q_{r-i}(x)e_i(y).$$

We let $\Gamma^{(k)}$ be the subring of $\Lambda \otimes \mathbb{Z}[y_1, \ldots, y_k]^{S_k}$ generated by the ϑ_r:

$$\Gamma^{(k)} = \mathbb{Z}[\vartheta_1, \vartheta_2, \vartheta_3, \ldots].$$

Set $\Theta(t) = \sum_{r\geq 0}\vartheta_r t^r$; we then have

$$\Theta(t) = \prod_i \frac{1+tx_i}{1-tx_i}\prod_{j=1}^{k}(1+y_jt) = E(x\,;t)H(x\,;t)E(y\,;t)$$

and hence

$$\Theta(t)\Theta(-t) = E(y\,;t)E(y\,;-t) = \sum_{m=0}^{2k}(-1)^m e_m(y^2),$$

where y^2 denotes $(y_1^2,\ldots,y_k^2)$. It follows that

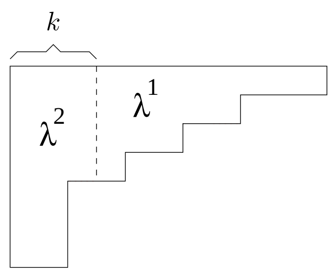
$$\sum_{r+s=d}(-1)^r\vartheta_r\vartheta_s = \begin{cases} 0 & \text{if } d \text{ is odd} \\ (-1)^d e_{d/2}(y^2) & \text{if } d \text{ is even.}\end{cases} \tag{21}$$

When $d = 2m > 2k$, equation (21) gives

$$\vartheta_m^2 = 2\sum_{i=1}^{m}(-1)^{i+1}\vartheta_{m+i}\vartheta_{m-i}. \tag{22}$$

For any integer sequence ρ, let $\vartheta_\rho = \prod_i \vartheta_{\rho_i}$, and define $q_\rho = \prod_i q_{\rho_i}$ and $e_\rho = \prod_i e_{\rho_i}$ similarly. We deduce from (22) that either λ is k-strict, or ϑ_λ is a $\mathbb{Z}$-linear combination of the ϑ_μ such that μ is k-strict and $\mu \succ \lambda$.

Definition 5.1. Given any k-strict partition λ, we obtain two partitions λ^1 and λ^2, with λ^1 strict, by cutting the Young diagram of λ into a disjoint union of two diagrams: λ^1 is the part of λ lying in columns $k+1$ and higher, while $\lambda^2 = \lambda \smallsetminus \lambda^1$.



For any partition λ, we have

$$\vartheta_\lambda(x\,;y) = \sum_\alpha q_{\lambda-\alpha}(x)e_\alpha(y),$$

the sum over all compositions α with $0 \le \alpha_i \le k$ for all i. If λ is k-strict, it follows that the homogeneous summand of ϑ_λ of lowest x-degree is equal to $q_{\lambda^1}(x)e_{\lambda^2}(y)$. The sets $\{q_\lambda(x) \mid \lambda \text{ strict}\}$ and $\{e_\lambda(y) \mid \lambda_i \le k,\ \forall i\}$ are linearly independent over $\mathbb{Z}$. We deduce that the ϑ_λ, λ k-strict, are linearly independent over $\mathbb{Z}$.

Equation (21) also implies that, for $m > k$, $\vartheta_{2m} \in \mathbb{Q}[\vartheta_1,\ldots,\vartheta_{2m-1}]$. By induction on m it follows that

$$\vartheta_{2m} \in \mathbb{Q}[\vartheta_1,\ldots,\vartheta_{2k},\vartheta_{2k+1},\vartheta_{2k+3},\ldots,\vartheta_{2m-1}]$$

for all $m > k$. Let $\Gamma^{(k)}_{\mathbb{Q}} = \Gamma^{(k)} \otimes \mathbb{Q}$. Then $\Gamma^{(k)}_{\mathbb{Q}}$ is generated by the ϑ_r with all $r > 2k$ odd.

We say that a partition λ is *k-odd* if all its parts which are greater than $2k$ are odd. For each $m \ge 0$, the number of k-odd partitions of m is equal to the number of k-strict partitions of m, because of the equality of generating functions

$$\sum_{\lambda\ k\text{-odd}} t^{|\lambda|} = \prod_{r=1}^{2k} \frac{1}{1-t^r} \prod_{r>k} \frac{1}{1-t^{2r-1}} = \prod_{r} \frac{1}{1-t^r} \prod_{r>k} (1-t^{2r})$$
$$= \prod_{r=1}^{k} \frac{1}{1-t^r} \prod_{r>k} (1+t^r) = \sum_{\lambda\ k\text{-strict}} t^{|\lambda|}\,.$$

We therefore have proved the following result.

Proposition 5.1. (i) *The ϑ_λ for λ k-strict form a $\mathbb{Z}$-basis of* $\Gamma^{(k)}$.

(ii) *The ϑ_λ for λ k-odd form a $\mathbb{Q}$-basis of* $\Gamma^{(k)}_{\mathbb{Q}}$.

5.2.

Definition 5.2. For any valid set of pairs $D \subset \Delta^\circ$ and integer sequence λ, define the polynomial $\Theta(D,\lambda)$ by the raising operator formula $\Theta(D,\lambda) = R^D \vartheta_\lambda$. Equivalently, we recursively set

$$\Theta(D,\lambda) = \sum_{\alpha} (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} \Theta(D,\mu+\alpha)\vartheta_{r-|\alpha|}\,,$$

where $\lambda = (\mu, r)$ has length ℓ and the sum is over all (D,ℓ)-compatible vectors $\alpha \in \mathbb{N}^{\ell-1}$. For any k-strict partition λ, the *theta polynomial* $\Theta_\lambda(x\,;y)$ is defined by $\Theta_\lambda = \Theta(\mathcal{C}(\lambda),\lambda) = R^\lambda \vartheta_\lambda$.

It follows from the raising operator definition that each Θ_λ is of the form

$$\Theta_\lambda = \vartheta_\lambda + \sum_{\mu \succ \lambda} c_{\lambda\mu}\vartheta_\mu$$

with coefficients $c_{\lambda\mu} \in \mathbb{Z}$. We deduce that we have

$$\Theta_\lambda = \vartheta_\lambda + \sum_{\mu \succ \lambda} c'_{\lambda\mu}\vartheta_\mu$$

where now the sum on the right is restricted to k-strict partitions $\mu \succ \lambda$. Since the latter form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$ by Proposition 5.1, it follows that the Θ_λ, as λ runs over k-strict partitions, form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$.

Let

$$\mathbb{H}(\mathrm{IG}_k) = \varprojlim \mathrm{H}^*(\mathrm{IG}(n-k,2n),\mathbb{Z})$$

be the stable cohomology ring of IG; that is, the inverse limit in the category of *graded* rings of the system

$$\cdots \leftarrow \mathrm{H}^*(\mathrm{IG}(n-k,2n),\mathbb{Z}) \leftarrow \mathrm{H}^*(\mathrm{IG}(n+1-k,2n+2),\mathbb{Z}) \leftarrow \cdots$$

From the presentation of $\mathrm{H}^*(\mathrm{IG}(n-k,2n),\mathbb{Z})$ given in [BKT1, Thm. 1.2], we deduce that $\mathbb{H}(\mathrm{IG}_k)$ is isomorphic to the polynomial ring $\mathbb{Z}[\sigma_1,\sigma_2,\ldots]$ modulo the relations

$$\sigma_m^2 + 2\sum_{i=1}^{m} (-1)^i \sigma_{m+i}\sigma_{m-i} = 0$$

for all $m > k$. Since the generators ϑ_r of $\Gamma^{(k)}$ satisfy (22), we have a homomorphism $\phi : \mathbb{H}(\mathrm{IG}_k) \to \Gamma^{(k)}$ sending σ_r to ϑ_r for each r. Theorem 1 implies that $\phi(\sigma_\lambda) = \Theta_\lambda$ for any k-strict partition λ. Since the Θ_λ form a basis of $\Gamma^{(k)}$, we conclude that ϕ is an isomorphism. This completes the proof of Theorem 2.

5.3. Consider the analogues of the polynomials ϑ_r when the $e_r(y)$ are replaced by complete symmetric functions $h_r(y)$. Define for each r a function $\widehat{\vartheta}_r = \widehat{\vartheta}_r(x\,;y)$ by

$$\widehat{\vartheta}_r = \sum_i q_{r-i}(x)h_i(y)$$

and set $\widehat{\Theta}(t) = \sum_{r\geq 0} \widehat{\vartheta}_r t^r$. We then have $\Theta(t)\widehat{\Theta}(-t) = 1$, or equivalently,

$$\sum_{r=0}^{n}(-1)^r\vartheta_r\widehat{\vartheta}_{n-r} = 0, \qquad n \geq 1. \tag{23}$$

The equations (23) imply that for any partitions λ and μ with $\mu \subset \lambda$,

$$\det\left(\vartheta_{\lambda_i-\mu_j+j-i}\right) = \det\left(\widehat{\vartheta}_{\lambda'_i-\mu'_j+j-i}\right),$$

and in particular,

$$\det\left(\vartheta_{\lambda_i+j-i}\right) = \det\left(\widehat{\vartheta}_{\lambda'_i+j-i}\right). \tag{24}$$

Here λ' is the partition conjugate to λ, i.e., $\lambda'_i = \#\{h \mid \lambda_h \geq i\}$ for all i.

Assume that $k > 0$, and let (1^r) denote the partition $(1,\ldots,1)$ of length r. We claim that for any $r \geq 1$, we have $\Theta_{(1^r)}(x\,;y) = \widehat{\vartheta}_r(x\,;y)$. Indeed, note that $\mathcal{C}(1^r) = \emptyset$, and hence (24) gives

$$\Theta_{(1^r)} = \prod_{i<j}(1-R_{ij})\,\vartheta_{(1^r)} = \det(\vartheta_{1+j-i})_{1\leq i,j\leq r} = \widehat{\vartheta}_r.$$

The polynomials $\widehat{\vartheta}_r = \Theta_{(1^r)}$ map to the Chern classes of the dual of the tautological subbundle $\mathcal{S} \to \mathrm{IG}$ under the isomorphism ϕ of §5.2.

5.4. We next introduce an analogue of the Schur S-functions in the ring $\Gamma^{(k)}$.

Definition 5.3. For any two finite integer sequences λ, μ, define the function $S^{(k)}_{\lambda/\mu} \in \Gamma^{(k)}$ by setting

$$S^{(k)}_{\lambda/\mu}(x\,;y) = \det(\vartheta_{\lambda_i-\mu_j+j-i}(x\,;y))_{i,j}.$$

Assume that λ and μ are two partitions. Then, arguing as in [M, I.5], the skew function $S^{(k)}_{\lambda/\mu}(x\,;y)$ is zero unless $\lambda_i \geq \mu_i$ for each i, in which case it depends only on the skew diagram $\lambda - \mu$. The functions $S_{\lambda/\mu}(x) := S^{(0)}_{\lambda/\mu}(x\,;y)$ are well known (see [M, III.8, Example 7] and [W, Sec. 2.7]). We also let

$$s_{\lambda'/\mu'}(y) = \det(e_{\lambda_i-\mu_j+j-i}(y))_{i,j}$$

denote the (ordinary) skew Schur polynomial in the variables y. We have that $s_{\lambda'/\mu'}(y) = 0$ unless $0 \leq \lambda_i - \mu_i \leq k$ for each i. The functions $S_{\lambda/\mu}(x)$ (respectively, $s_{\lambda'/\mu'}(y)$) are known to be linear combinations of Schur Q-functions $Q_\nu(x)$ (respectively, Schur S-polynomials $s_{\nu'}(y)$) with positive integer coefficients.

Proposition 5.2. *For any partitions λ, μ with $\mu \subset \lambda$, we have*

$$S^{(k)}_{\lambda/\mu}(x\,;y) = \sum_\nu S_{\lambda/\nu}(x)s_{\nu'/\mu'}(y) = \sum_\nu S_{\nu/\mu}(x)s_{\lambda'/\nu'}(y) \tag{25}$$

summed over all partitions ν such that $\mu \subset \nu \subset \lambda$.

Proof. Let $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \ldots)$ be another countably infinite set of variables and define the ring $\widetilde{\Lambda} = \mathbb{Z}[e_1(\widetilde{x}), e_2(\widetilde{x}), \ldots] \otimes_{\mathbb{Z}} \mathbb{Z}[e_1(y), \ldots, e_k(y)]$. Consider the ring homomorphism $\epsilon : \widetilde{\Lambda} \to \Gamma^{(k)}$ defined by sending $e_i(\widetilde{x})$ to $q_i(x)$ and $e_j(y)$ to $e_j(y)$. According to [M, I.(5.10)], the identity

$$s_{\lambda'/\mu'}(\widetilde{x}, y) = \sum_{\nu} s_{\lambda'/\nu'}(\widetilde{x}) s_{\nu'/\mu'}(y) = \sum_{\nu} s_{\nu'/\mu'}(\widetilde{x}) s_{\lambda'/\nu'}(y)$$

holds in $\widetilde{\Lambda}$. This is mapped to (25) under the homomorphism ϵ. □

The definition of $S_\lambda^{(k)}$ implies that $S_\lambda^{(k)} = \vartheta_\lambda + \sum_{\mu \succ \lambda} d_{\lambda\mu} \vartheta_\mu$ for some integers $d_{\lambda\mu}$, and therefore that the set of $S_\lambda^{(k)}$ for λ k-strict forms another $\mathbb{Z}$-basis of $\Gamma^{(k)}$. The next result follows immediately from the definitions.

Proposition 5.3. *For any k-strict partition λ, we have*

$$\Theta_\lambda(x\,; y) = \prod_{(i,j) \in \mathcal{C}(\lambda)} (1 - R_{ij} + R_{ij}^2 - \cdots)\, S_\lambda^{(k)}(x\,; y).$$

5.5. In this section, we give the proof of Theorem 3. Let λ be a k-strict partition. Note that if $\lambda_i + \lambda_j \leq 2k + j - i$ for all $i < j$, then $\mathcal{C}(\lambda) = \emptyset$, and hence $\Theta_\lambda = S_\lambda^{(k)}$. Part (a) of the theorem then follows by setting $\mu = 0$ in (25). Observe that in this case we also have

$$\Theta_\lambda(x\,; y) = \sum_{\mu,\nu} c_{\mu\nu}^\lambda S_\mu(x) s_{\nu'}(y) = \sum_{\mu \subset \lambda} S_{\lambda/\mu}(x) s_{\mu'}(y),$$

where $c_{\mu\nu}^\lambda$ is a Littlewood-Richardson coefficient.

Suppose now that the partition λ is such that $\lambda_i + \lambda_j > 2k + j - i$ for all $i < j \leq \ell(\lambda)$; in particular, λ is *strict*. For $\ell = \ell(\lambda)$ and $\epsilon_\ell = (0, 1, \ldots, \ell - 1)$, we define the shifted skew shape

$$\mathcal{S}(\lambda/\mu) = (\lambda + \epsilon_\ell)/(\mu + \epsilon_\ell)$$

for any strict partition $\mu \subset \lambda$.

We claim that

$$\Theta_\lambda = \sum_\alpha Q_{\lambda-\alpha}(x) e_\alpha(y), \tag{26}$$

where the sum runs over all compositions α with $0 \leq \alpha_i \leq k$ for each i. Indeed, for any raising operator R, we have

$$R\,\vartheta_\lambda(x\,; y) = \vartheta_{R\lambda}(x\,; y) = \sum_\alpha e_\alpha(y)\, q_{R\lambda-\alpha}(x) = \sum_\alpha e_\alpha(y)\, R\, q_{\lambda-\alpha}(x).$$

It follows that

$$\Theta_\lambda = \sum_\alpha e_\alpha(y)\, R^\lambda\, q_{\lambda-\alpha}(x).$$

For a fixed composition α, the effect of the Schur Pfaffian operator R^λ on the term $q_{\lambda-\alpha}(x)$ is to convert it to the term $Q_{\lambda-\alpha}(x)$. Equation (26) follows.

Since λ is strict and $\lambda_{\ell-1} + \lambda_\ell > 2k+1$, we see that $\lambda_i > \alpha_i$ for all compositions α indexing the sum (26) and every i except possibly $i = \ell$. If $\lambda_\ell < \alpha_\ell$ then $Q_{\lambda-\alpha} = 0$; therefore all non-vanishing terms in the sum are indexed by (nonnegative) compositions. If any index has a repeated part, the Q-function again vanishes. Let

$\delta_\ell = (\ell-1, \ell-2, \ldots, 1, 0)$ and $b = \delta_\ell + \epsilon_\ell = (\ell-1)^\ell$. It follows that we may rewrite (26) as

$$\Theta_\lambda = \sum_\mu \sum_{w \in S_\ell} (-1)^w Q_\mu(x) e_{\lambda - w(\mu)}(y)$$

summed over strict partitions μ of length at most ℓ. For each $w \in S_n$, we have

$$\lambda - w(\mu) = \lambda + b - w(\mu + b) = (\lambda + \epsilon_\ell) + \delta_\ell - w((\mu + \epsilon_\ell) + \delta_\ell).$$

Therefore

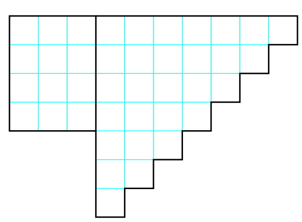
$$\Theta_\lambda = \sum_\mu Q_\mu(x) \sum_{w \in S_\ell} (-1)^w e_{(\lambda+\epsilon_\ell)+\delta_\ell - w((\mu+\epsilon_\ell)+\delta_\ell)}(y),$$

the sum over all strict partitions $\mu \subset \lambda$. The latter sum is equal to the one in the statement of the theorem. Note that the skew Schur function $s_{\mathcal{S}(\lambda/\mu)'}(y)$ vanishes unless μ has length at least $\ell(\lambda) - 1$.

6. Schubert Polynomials for Isotropic Grassmannians

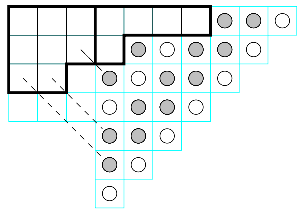
6.1. The polynomials $\Theta_\lambda(x\,;y)$ fall within the Billey-Haiman theory of type C Schubert polynomials $\mathfrak{C}_w(x, z)$. We will prove and discuss this in detail in this section. Let W_n be the hyperoctahedral group of signed permutations on the set $\{1, \ldots, n\}$, and $W_\infty = \cup_n W_n$. The group W_∞ is generated by the simple transpositions $s_i = (i, i+1)$ for $i > 0$, and the sign change $s_0(1) = \overline{1}$. The elements of W_n index the Schubert classes in the cohomology ring of the flag variety Sp_{2n}/B, which includes $\mathrm{H}^*(\mathrm{IG}(n-k, 2n), \mathbb{Z})$ as a subring. In particular every k-strict partition $\lambda \in \mathcal{P}(k, n)$ corresponds to a *Grassmannian element* $w_\lambda \in W_n$, which we proceed to describe; the reader may consult [T1, §4] for more details.

The elements of $\mathcal{P}(k, n)$ are exactly the k-strict partitions whose diagrams fit inside a shape Π, obtained by attaching an $m \times k$ rectangle to the left side of a staircase partition with n rows. When $n = 7$ and $k = 3$, this looks as follows.

$\Pi =$ 

The signed permutation $w_\lambda = (w_1, \ldots, w_n)$ has a unique descent at k, that is, $w(i) < w(i+1)$ whenever $i \neq k$. For $\lambda \in \mathcal{P}(k, n)$ we let λ^1 be the strict partition formed by the boxes of λ in columns $k+1$ through $k+n$. The negative entries of w_λ are then given by the parts of λ^1.

The boxes of the staircase partition which are outside λ are organized into south-west to north-east diagonals. The k diagonals which are k-related to one of the bottom boxes in the first k columns of λ are called *related*; the remaining diagonals are non-related. The first k entries of w_λ are the lengths of the related diagonals, and the last $n - k - \ell_k(\lambda)$ entries are the lengths of the non-related diagonals. For example, the partition $\lambda = (7,4,2) \in \mathcal{P}(3,7)$ results in the element $w_\lambda = 256\overline{4}\overline{1}37$.

$\lambda =$ 

6.2. A sequence $a = (a_1, \ldots, a_m)$ is called *unimodal* if for some r with $0 \leq r \leq m$, we have

$$a_1 > a_2 > \cdots > a_r < a_{r+1} < \cdots < a_m.$$

Let $w \in W_\infty$ and λ be a Young diagram with r rows such that $|\lambda| = \ell(w)$. A *Kraśkiewicz tableau* [Kr] for w of shape λ is a filling T of the boxes of λ with nonnegative integers in such a way that (a) if t_i is the sequence of entries in the i-th row of T, reading from left to right, then the row word $t_r \ldots t_1$ is a reduced word for w; and (b) for each i, t_i is a unimodal subsequence of maximum length in $t_r \ldots t_{i+1} t_i$.

For each $w \in W_\infty$ one has a *type C Stanley symmetric function* $F_w(x)$, which is a positive linear combination of Schur Q-functions. There exist several combinatorial interpretations for the coefficients in this expression. We will use the following result of Lam [L]:

$$F_w(x) = \sum_\lambda e_w^\lambda \, Q_\lambda(x) \tag{27}$$

where e_w^λ equals the number of Kraśkiewicz tableaux for w of shape λ.

Example 6.1. Suppose that $k = 0$ and $w = w_\lambda$ is a maximal (Lagrangian) Grassmannian element corresponding the strict partition λ. In this case there is only one Kraśkiewicz tableau for w, which has shape λ, and is given as in the following example, for $\lambda = (6,5,2)$:

$$\begin{matrix} 5\ 4\ 3\ 2\ 1\ 0 \\ 4\ 3\ 2\ 1\ 0 \\ 1\ 0. \end{matrix}$$

It follows that $F_{w_\lambda}(x) = Q_\lambda(x)$, for all such λ.

Following Billey and Haiman, each $w \in W_\infty$ indexes a type C Schubert polynomial $\mathfrak{C}_w(x,z)$. Here $z = (z_1, z_2, \ldots)$ is another infinite set of variables and each $\mathfrak{C}_w$ is a polynomial in the ring $A = \mathbb{Z}[q_1(x), q_2(x), \ldots; z_1, z_2, \ldots]$. The $\mathfrak{C}_w$ for $w \in W_\infty$ form a $\mathbb{Z}$-basis of A, and their algebra agrees with the Schubert calculus on symplectic flag varieties Sp_{2n}/B, when n is sufficiently large. According to [BH, Thm. 3], for any $w \in W_n$ we have

$$\mathfrak{C}_w(x,z) = \sum_{uv=w} F_u(x) \mathfrak{S}_v(z), \tag{28}$$

the sum over all reduced factorizations $uv = w$ in W_n (i.e., such that $\ell(u) + \ell(v) = \ell(w)$) with $v \in S_n$. Here $\mathfrak{S}_v(z)$ denotes the type A Schubert polynomial of Lascoux and Schutzenberger [LS].

6.3. We will show next that the theta polynomial ϑ_r agrees with the Billey-Haiman polynomial indexed by the Grassmannian permutation $w_{(r)} \in W_n$ corresponding to $\lambda = r$, for $1 \leq r \leq n+k$. It is easy to see that there is a unique reduced word for any such element; these $n+k$ words are listed below.

$$s_k,\ s_{k-1}s_k,\ \ldots,\ s_1\cdots s_k,\ s_0s_1\cdots s_k,\ s_1s_0s_1\cdots s_k,\ \ldots,\ s_{n-1}\cdots s_1s_0s_1\cdots s_k.$$

Set $v_i = s_is_{i+1}\ldots s_k$. For any reduced factorization $w_{(r)} = uv$ with $v \in S_n$, it is immediate that $v = v_i$ for some $i > 0$. The type A Schubert polynomial for v_i is exactly the elementary symmetric polynomial $e_{k+1-i}(z_1,\ldots,z_k)$, for $1 \leq i \leq k$. Moreover, if $u_i = w_{(r)}v_i^{-1}$, then (27) implies that $F_{u_i}(x) = q_{r+i-k-1}(x)$. We conclude from (28) that

$$\mathfrak{C}_{w_{(r)}}(x,z) = \sum_{j\geq 0} q_{r-j}(x_1,x_2,\ldots)e_j(z_1,\ldots,z_k) = \vartheta_r(x,z),$$

as required. Since both the theta and Billey-Haiman polynomials form a basis for the ring that they span, we deduce the following result.

Corollary 6.1. *The ring $\Gamma^{(k)}$ of theta polynomials is a subring of the ring of Billey-Haiman Schubert polynomials of type C. For every k-strict partition λ, we have $\Theta_\lambda(x\,;y) = \mathfrak{C}_{w_\lambda}(x,y)$.*

We have shown that for every k-strict λ, there is a unique expression

$$\Theta_\lambda(x\,;y) = \sum_{uv=w_\lambda} F_u(x)\mathfrak{S}_v(y), \tag{29}$$

the sum over all reduced factorizations $uv = w_\lambda$ in W_∞ with $v \in S_\infty$. The right factor v in any such factorization must be a Grassmannian permutation with unique descent at k. Hence each Schubert polynomial $\mathfrak{S}_v(y)$ will be symmetric in y, and therefore equal to a Schur polynomial $s_{\nu'}(y)$ for some partition ν; in fact, one checks easily that $\nu \subset \lambda^2$. The coefficient of $Q_\mu(x)s_{\nu'}(y)$ in (29) is equal to the number of Kraśkiewicz tableaux for $w_\lambda v^{-1}$ of shape μ. This completes the proof of Theorem 4.

Corollary 6.2. a) *For any k-strict partition λ, the homogeneous summand of $\Theta_\lambda(x\,;y)$ of highest x-degree is the type C Stanley symmetric function $F_{w_\lambda}(x)$, and satisfies $F_{w_\lambda}(x) = R^\lambda q_\lambda(x)$.*

b) *The homogeneous summand of $\Theta_\lambda(x\,;y)$ of lowest x-degree is $Q_{\lambda^1}(x)\,s_{(\lambda^2)'}(y)$.*

Proof. Part (a) is deduced by setting $y = 0$ in (29) and also in the raising operator expression $\Theta_\lambda(x\,;y) = R^\lambda\vartheta_\lambda(x\,;y)$. For part (b), notice that there is a unique permutation $v \in S_\infty$ of maximal length such that w_λ has a reduced factorization $w_\lambda = uv$. In this case u is the maximal Grassmannian Weyl group element corresponding to the strict partition λ^1. The result now follows, using Example 6.1. □

Example 6.2. Let $k = 1$ and $\lambda = (3,2,1)$, with corresponding Weyl group element $w_\lambda = 4\overline{2}\overline{1}3 \in W_4$. Then we have

$$\Theta_{321} = (Q_{42} + Q_{321}) + (Q_{41} + 2\,Q_{32})\,s_{1'} + 2\,Q_{31}\,s_{11'} + Q_{21}\,s_{111'} \tag{30}$$

(with the variables x and y omitted). The Kraśkiewicz tableaux which correspond to the summands on the right hand side of (30) are given in the following table.

u	v	Kraśkiewicz tableaux for u
$4\overline{2}\overline{1}3$	1234	3201 01 321 10 0
$\overline{2}4\overline{1}3$	2134	3102 0 320 01 302 01
$\overline{2}\overline{1}43$	3124	310 0 103 0
$\overline{2}\overline{1}34$	4123	10 0

6.4. We remark that to obtain polynomials that multiply like the Schubert classes on the orthogonal Grassmannians $\mathrm{OG}(n-k,2n+1)$, one simply uses the $2^{-\ell_k(\lambda)}\Theta_\lambda$ for all k-strict λ. These polynomials agree with the Billey-Haiman Schubert polynomials of type B indexed by Grassmannian elements w_λ. For the even orthogonal Grassmannians $\mathrm{OG}(n-k,2n)$, both the Giambelli formula and the corresponding family of polynomials are more involved; we plan to develop this theory elsewhere.

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