Math 406 – Fall 2009 – Harry Tamvakis

PROBLEM SET 1 – Due September 10, 2009

Reading for this week: Appendix A (Induction), and Section 1 (Divisibility and the Division Algorithm).

Problems

From the textbook: Section 1, Problems #7, 12, 15, Appendix A, Problems #4, 7, 8, 11. In addition, do the following problems:

A1) Suppose that r is a real number with $r \neq 1$. Show that for any natural number n, we have

$$a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1}.$$

A2) Suppose that the numbers a_n are defined recursively by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Use the strong induction principle to show that $a_n < 2^n$ for every natural number n.

A3) a) Prove that the square of any integer is either of the form $3k$ or $3k+1$.

b) Prove that the cube of any integer has one of the forms: $9k$, $9k + 1$, or $9k + 8$.

c) Show that there exist integers which cannot be written as a sum of three cubes. For example, verify that there do not exist integers x, y, and z (possibly negative) such that $x^3 + y^3 + z^3 = 5$.

Extra Credit Problems.

EC1) a) Given the 2×2 matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, use induction to prove that, for all $n \geqslant 1$, we have

$$A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix},$$

where f_n represents the n-th Fibonacci number, with the convention that $f_0 = 0$.

b) Use part (a) to prove the identity

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n.$$

EC2) Prove that for every natural number n, the expression

$$(2+\sqrt{3})^n + (2-\sqrt{3})^n$$

is equal to a natural number.