Math 406 – Fall 2023 – Harry Tamvakis

PROBLEM SET 1 – Due September 7, 2023

Reading for this week: Appendix A (Proof by Induction, pp. 205-209).

Problems

From the textbook: Appendix A, Problems #1, 2, 4, 7, 11, 14. In addition, do the following problems:

A1) Suppose that \( x \) is variable. Show that for any natural number \( n \), we have the identity

\[
x^1 + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.
\]

A2) Suppose that the numbers \( a_n \) are defined recursively by \( a_1 := 1 \), \( a_2 := 2 \), \( a_3 := 3 \), and \( a_n = a_{n-1} + a_{n-2} + a_{n-3} \) for all \( n \geq 4 \). Use the strong induction principle to show that \( a_n < 2^n \) for every natural number \( n \).

Extra Credit Problems.

EC1) (a) Given the \( 2 \times 2 \) matrix \( A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \), use induction to prove that, for all integers \( n \geq 1 \), we have

\[
A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.
\]

Here \( F_n \) is the \( n \)-th Fibonacci number, with the convention that \( F_0 := 0 \).

(b) Use part (a) to prove the identity

\[
F_{n+1}F_{n-1} - F_n^2 = (-1)^n
\]

for any \( n \geq 1 \).

EC2) Prove that for every integer \( n \geq 1 \), the expression

\[
(2 + \sqrt{3})^n + (2 - \sqrt{3})^n
\]

is equal to a natural number.