

Math 406 – Fall 2009 – Harry Tamvakis

Answer Key for PROBLEM SET 1 – Due September 10, 2009

Grader - Allen Gehret

Problems

S1.7 Prove that if $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.

Proof. If $a \mid b$ and $b \mid a$ then there exists $c, d \in \mathbb{Z}$ such that $b = ac$ and $a = db$. Substituting the first equality into the second we get $a = d(ac) = (cd)a$. Thus $cd = 1$ so either $c = d = 1$ or $c = d = -1$.

(First case) If $c = d = 1$ then $a = 1 \cdot b = b$.

(Second case) If $c = d = -1$ then $a = -1 \cdot b = -b$. □

S1.12 Prove: If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

Proof. If $a \mid b$ and $c \mid d$ then there exists $f, g \in \mathbb{Z}$ such that $b = fa$ and $d = gc$. Multiplying these two equations together gives us $bd = (fa)(gc) = (fg)ac$. Since there exists a $k = fg \in \mathbb{Z}$ such that $bd = k(ac)$, by definition we conclude that $ac \mid bd$. □

S1.15 (a) If $x^2 + ax + b = 0$ has an integer root, show that it divides b .

Proof. If there exists an $x \in \mathbb{Z}$ such that $x^2 + ax + b = 0$, then that same x satisfies the equation $b = (-x - a)x$ (since all we did was solve for b and factor the right hand side). Note that if $x, a \in \mathbb{Z}$ then $(-x - a) \in \mathbb{Z}$. Let $k = (-x - a)$. Since there exists a $k \in \mathbb{Z}$ such that $b = kx$ then by definition $x \mid b$. □

S1.15 (b) If $x^2 + ax + b = 0$ has a rational root, show that it is in fact an integer.

Proof. Let $\frac{p}{q} \in \mathbb{Q}$ be a root of the equation such that $p, q \in \mathbb{Z}$, $q > 0$ and $(p, q) = 1$. Note that

$$\begin{aligned} \left(\frac{p}{q}\right)^2 + a\left(\frac{p}{q}\right) + b &= 0 \\ \frac{p^2 + apq + bq^2}{q^2} &= 0 \\ p^2 + apq + bq^2 &= q^2 \\ (b-1)q^2 + (ap)q + p^2 &= 0 \end{aligned}$$

This last line is a quadratic equation with integer coefficients which has q as an *integer* root. Applying part (a) tells us that $q \mid p^2$. Corollary 1 tells us that $q \mid p$ since $(q, p) = 1$. Since $q \mid p$ and $q \mid q$, $q \mid (p, q) = 1$ so $q = 1$. Thus $\frac{p}{q} = p \in \mathbb{Z}$. \square

Ap A.4 Prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} = 1 - \frac{1}{n}$$

for $n = 2, 3, \dots$

Proof. (Base case, $n=2$) In this case it is clear that $\frac{1}{1 \cdot 2} = 1 - \frac{1}{2} = 1 - \frac{1}{2}$.

(Inductive step) For the inductive hypothesis, assume that the statement is true for $n = k$, i.e., assume that

$$\frac{1}{1 \cdot 2} + \cdots + \frac{1}{(k-1)k} = 1 - \frac{1}{k}.$$

We now desire to prove the statement true for $n = k + 1$. Note that

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(k-1)k} + \frac{1}{k(k+1)} &= \left(1 - \frac{1}{k}\right) + \frac{1}{k(k+1)} \\ &= 1 + \left(\frac{-(k+1)}{k(k+1)} + \frac{1}{k(k+1)}\right) \\ &= 1 + \left(\frac{-k-1+1}{k(k+1)}\right) \\ &= 1 - \frac{1}{k+1} \end{aligned}$$

By the principle of mathematical induction, the statement is true for all $n = 2, 3, \dots$ \square

Ap A.7 Suppose that $a_0 = a_1 = 1$ and $a_{n+1} = a_n + 2a_{n-1}, n = 1, 2, \dots$. Prove by induction that $a_n = \frac{2^{n+1} + (-1)^n}{3}$.

Proof. (Base cases, a_0 and a_1) Note that $\frac{2^{0+1} + (-1)^0}{3} = \frac{2+1}{3} = 1 = a_0$ and that $\frac{2^{1+1} + (-1)^1}{3} = \frac{4-1}{3} = 1 = a_1$.

(Inductive Step) For $n \geq 2$ assume that the equation $a_k = \frac{2^{k+1} + (-1)^k}{3}$ is true for all $0 \leq k < n$. Note that

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} \\ &= \frac{2^n + (-1)^{n-1}}{3} + 2 \left(\frac{2^{n-1} + (-1)^{n-2}}{3} \right) \\ &= \frac{2^n + (-1)^{n-1} + 2^n + 2(-1)^{n-2}}{3} \\ &= \frac{2^{n+1} + (-1 + 2)(-1)^{n-2}}{3} \\ &= \frac{2^{n+1} + (-1)^{n-1}}{3} \end{aligned}$$

By the principle of strong mathematical induction, we can conclude that the statement is true for all $n = 1, 2, \dots$ \square

Ap A.8 Suppose that $a_1 = a_2 = 1$ and $a_{n+1} = 3a_n + a_{n-1}$. Prove that $(a_n, a_{n+1}) = 1, n = 1, 2, \dots$

Proof. (Base Case, $n = 1$) Clearly $(a_1, a_2) = (1, 1) = 1$.

(Inductive step) Assume that (a_k, a_{k-1}) is true for some $k \geq 1$. We want to show that the statement is true for $k + 1$. By Lemma 3 this is clear since $a_{k+1} = 3a_k + a_{k-1}$ implies that $(a_{k+1}, a_k) = (a_k, a_{k-1}) = 1$ (just make the assignment $a = a_{k+1}, q = 3, b = a_k$ and $r = a_{k-1}$). \square

Ap A.11 Prove by induction that $n(n+1)(n+2)$ is divisible by 6 for $n = 1, 2, \dots$

Proof. (Base cases $n = 1, 2, 3$) Note that $1(1+1)(1+2) = 2 \cdot 3 = 6$, $2(2+1)(2+2) = 2 \cdot 3 \cdot 4 = 24 = 4 \cdot 6$ and $3(3+1)(3+2) = 3 \cdot 4 \cdot 5 = 60 = 10 \cdot 6$.

(Inductive step) Assume that for $n > 2$ that the statement is true for all $0 < k \leq n$. Note that

$$(n+1)(n+2)(n+3) = n(n+1)(n+2) + 3(n+1)(n+2)$$

Since $6 \mid n(n+1)(n+2)$ (by the inductive hypothesis) and $3 \mid 3, 2 \mid (n+1)(n+2)$ implies that $6 \mid 3(n+1)(n+2)$, we conclude that

$$6 \mid (n+1)(n+2)(n+3).$$

By the principle of mathematical induction the statement is true for all $n = 1, 2, \dots$ \square

A1) Suppose that r is a real number with $r \neq 1$. Show that for any natural number n , we have

$$a + ar + ar^2 + \dots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1}.$$

Proof. (Base case, $n = 1$) Note that $a + ar = \frac{a(r+1)(r-1)}{(r-1)} = \frac{a(r^2-1)}{r-1}$.

(Inductive step) For our inductive hypothesis we will assume that the statement is true for some $n \geq 1$. Now note that

$$\begin{aligned} a + ar + \dots + ar^n + ar^{n+1} &= \frac{a(r^{n+1} - 1)}{r - 1} + ar^{n+1} \\ &= \frac{a(r^{n+1} - 1) + ar^{n+1}(r - 1)}{r - 1} \\ &= \frac{a(r^{n+1} - 1 + r^{n+2} - r^{n+1})}{r - 1} \\ &= \frac{a(r^{n+2} - 1)}{r - 1}. \end{aligned}$$

Thus the statement is true for $n + 1$.

By the principle of mathematical induction we conclude that the statement is true for all natural numbers n . \square

A2) Suppose that the numbers a_n are defined recursively by $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for all $n \geq 4$. Use the strong induction principle to show that $a_n < 2^n$ for every natural number n .

Proof. (Base cases, $n = 1, 2, 3$) Note that $a_1 = 1 < 2^1 = 2$, $a_2 = 2 < 2^2 = 4$ and $a_3 = 3 < 2^3 = 8$.

(Inductive step) We will assume that $a_k < 2^k$ for all $0 < k < n$ for some $n \geq 4$. Note that

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} + a_{n-3} \\ &< 2^{n-1} + 2^{n-2} + 2^{n-3} \\ &< 2^{n-1} + 2^{n-2} + 2 \cdot 2^{n-3} \\ &= 2^{n-1} + 2 \cdot 2^{n-2} \\ &= 2 \cdot 2^{n-1} \\ &= 2^n. \end{aligned}$$

Thus $a_n < 2^n$.

By the principle of strong mathematical induction, we can conclude that $a_n < 2^n$ for all natural numbers n . \square

A3) a) Prove that the square of any integer is either of the form $3k$ or $3k + 1$.

Proof. By the division algorithm, every integer is of the form $3k, 3k + 1$ or $3k + 2$ for some $k \in \mathbb{Z}$. Thus we have 3 cases to check.

($3k$ case) Note that $(3k)^2 = 3(3k^2)$ is of the form $3k'$ where $k' = 3k^2$.

($3k + 1$ case) Note that $(3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ is of the form $3k' + 1$ where $k' = 3k^2 + 2k$.

($3k + 2$ case) Note that $(3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ is of the form $3k' + 1$ where $k' = 3k^2 + 4k + 1$.

We conclude that the square of any integer is either of the form $3k$ or $3k + 1$. \square

b) Prove that the cube of any integer has one of the forms: $9k, 9k + 1$, or $9k + 8$.

Proof. By the division algorithm, every integer is of the form $3k, 3k + 1$ or $3k + 2$ for some $k \in \mathbb{Z}$. Thus we have 3 cases to check.

($3k$ case) Note that $(3k)^3 = 9(3k^3)$ is of the form $9k'$ where $k' = 3k^3$.

($3k+1$ case) Note that $(3k+1)^3 = 27k^3 + 27k^2 + 9k + 8 = 9(3k^3 + 3k^2 + k) + 8$ is of the form $9k' + 8$ where $k' = 3k^3 + 3k^2 + k$.

($3k+2$ case) Note that $(3k+2)^3 = 27k^3 + 54k^2 + 36k + 8 = 3(3k^3 + 6k^2 + 4k) + 8$ is of the form $9k' + 8$ where $k' = 3k^3 + 6k^2 + 4k$.

We conclude that the cube of any integer is either of the form $9k$, $9k + 1$ or $9k + 8$. \square

c) Show that there exist integers which cannot be written as a sum of three cubes. For example, verify that there do not exist integers x , y , and z (possibly negative) such that $x^3 + y^3 + z^3 = 5$.

Proof. By using part (b) and some calculation, one can show that the sum of three cubes can be in one of the following forms: $9k$, $9k + 1$, $9k + 2$, $9k + 3$, $9k + 6$, $9k + 7$, and $9k + 8$. Thus anything of the form $9k + 4$ or $9k + 5$ can never be written as the sum of three cubes. \square

Extra Credit Problems.

EC1) a) Given the 2×2 matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, use induction to prove that, for all $n \geq 1$, we have

$$A^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix},$$

where f_n represents the n -th Fibonacci number, with the convention that $f_0 = 0$.

Proof. (Base case, $n = 1$) Note that $f_2 = f_1 = 1$ (by definition of the Fibonacci numbers) and $f_0 = 0$.

(Inductive step) For our inductive hypothesis, we will assume that the statement is true for some $n \geq 1$ and we will try to show that it is true for

$n + 1$.

$$\begin{aligned} A^{n+1} &= A^n A \\ &= \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} f_{n+1} + f_n & f_{n+1} \\ f_n + f_{n-1} & f_n \end{pmatrix} \\ &= \begin{pmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{pmatrix}. \end{aligned}$$

By the principle of mathematical induction we conclude that the statement is true for all $n \geq 1$. \square

b) Use part (a) to prove the identity

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n.$$

Proof. Using basic properties of determinants, note that

$$\begin{aligned} \det A^n &= \det \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \\ \left(\det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)^n &= (f_{n+1}f_{n-1} - f_n^2) \\ (-1)^n &= f_{n-1}f_n - f_n^2 \end{aligned}$$

\square

EC2) Prove that for every natural number n , the expression

$$(2 + \sqrt{3})^n + (2 - \sqrt{3})^n$$

is equal to a natural number.

Proof. First note that

$$\begin{aligned} (x - (2 + \sqrt{3})) (x - (2 - \sqrt{3})) &= x^2 - (2 - \sqrt{3})x - (2 + \sqrt{3})x + 1 \\ &= x^2 - 4x + 1. \end{aligned}$$

Thus $(2 + \sqrt{3})$ and $(2 - \sqrt{3})$ are the two roots of the quadratic equation $x^2 - 4x + 1$. We now proceed with the proof.

(Base cases, $n = 1, 2$). Note that $(2 + \sqrt{3}) + (2 - \sqrt{3}) = 4 \in \mathbb{N}$ and that $(2 + \sqrt{3})^2 + (2 - \sqrt{3})^2 = 4 + 2\sqrt{3} + 3 + 4 - 2\sqrt{3} - 3 = 8 \in \mathbb{N}$.

(Induction Step) For $n > 2$ assume for all $0 < k < n$ that $(2 + \sqrt{3})^k + (2 - \sqrt{3})^k \in \mathbb{N}$. Now note that

$$\begin{aligned}
 (2 + \sqrt{3})^n + (2 - \sqrt{3})^n &= 4(2 + \sqrt{3})^{n-1} - (2 + \sqrt{3})^{n-2} \\
 &\quad + 4(2 - \sqrt{3})^{n-1} - (2 - \sqrt{3})^{n-2} \\
 &= 4 \left[(2 + \sqrt{3})^{n-1} + (2 - \sqrt{3})^{n-1} \right] \\
 &\quad - \left[(2 + \sqrt{3})^{n-2} + (2 - \sqrt{3})^{n-2} \right] \\
 &= 4k - l
 \end{aligned}$$

for some $k, l \in \mathbb{N}$. If $k > l$ we would be done since $4k - l \in \mathbb{N}$ in that case. This is the case, however, since

$$\begin{aligned}
 k &= (2 + \sqrt{3})^{n-1} + (2 - \sqrt{3})^{n-1} \\
 &> (2 + \sqrt{3})^{n-1} \\
 &= (2 + \sqrt{3})^{n-2} (2 + \sqrt{3}) \\
 &= 2(2 + \sqrt{3})^{n-2} + \sqrt{3}(2 + \sqrt{3})^{n-2} \\
 &> (2 + \sqrt{3})^{n-2} + (2 + \sqrt{3})^{n-2} \\
 &> (2 + \sqrt{3})^{n-2} + (2 - \sqrt{3})^{n-2} \\
 &= l.
 \end{aligned}$$

□