Math 406 – Fall 2009 – Harry Tamvakis

PROBLEM SET 3 – Due September 24, 2009

Reading for this week: Section 3 and whatever we cover from Section 4.

Problems

From the textbook: Section 3, Problems #2, 5, 6, Section 4, Problems #2, 3, 4, 6. In addition, do the following problems:

A1) Let $\tau(n)$ be the number of positive divisors of n. Show that $\tau(n) = \tau(n+1) = \tau(n+2) = \tau(n+3)$ if $n = 3655$.

A2) Prove that the integer $53^{103} + 103^{53}$ is divisible by 39, and that $111^{333} + 333^{111}$ is divisible by 7.

A3) For each $n \geq 1$, use congruence theory to establish each of the following statements.

a) $7 \mid 5^{2n} + 3 \cdot 2^{5n-2}$

b) $13 \mid 3^{n+2} + 4^{2n+1}$

c) $27 \mid 2^{5n+1} + 5^{n+2}$.

A4) Let a be an odd integer. Use induction on n to prove that for any $n \geq 1$,

$$a^{2^n} \equiv 1 \pmod{2^{n+2}}.$$

A5) Prove that, for any natural numbers m and n, the number $3^m + 3^n + 1$ is never a perfect square. [Hint: Work modulo 8].

Extra Credit Problems.

EC1) A certain prison row has 1000 cells, each holding one prisoner. A jailer, carrying out the terms of a partial amnesty, unlocked every cell in the prison row. Next he locked every second cell. Then he turned the key in every third cell, locking those cells which were open and opening those cells which were locked. He continued in this way, on the n-th trip turning the key in every n-th cell, for $n = 1, 2, \ldots, 1000$. At the end of the process, those prisoners whose cells remained open were allowed to go free. How many prisoners were set free? Explain your reasoning.

EC2) A positive integer is called *polite* if it can be represented as a sum of two or more consecutive natural numbers. For example, 7 and 22 are polite since $7 = 3 + 4$ and $22 = 4 + 5 + 6 + 7$, while 2 is impolite. Prove that the only impolite positive integers are the powers of 2, that is, 1, 2, 4, 8, 16, ...