Math 406 – Fall 2009 – Harry Tamvakis

PROBLEM SET 3 – Due September 24, 2009

Grader - Allen Gehret

Problems

S3.2) Find all the integer solutions of $2x+y=2$, $3x-4y=0$, and $15x+18y=17$.

Proof. Since $(2,1)=1$ and $(1,0)$ is a particular solution and by Theorem 1 on pg.25, every solution is of the form $x=1+t, y=-2t$ for $t\in\mathbb{Z}$.

Since $(3,4)=1$ and $(4,3)$ is a particular solution and by Theorem 1 on pg.25, every solution is of the form $x=4-4t, y=3-3t$ for $t\in\mathbb{Z}$.

Since $(15,18)=3$ and 3 does not divide 17, we conclude by Lemma 2 on pg.22 that $15x+18y=17$ has no integer solutions. □

S3.5) Find all the positive solutions in integers of

$$\begin{aligned} x+y+z &= 31, \\ x+2y+3z &= 41. \end{aligned}$$

Proof. Subtracting the top line from the bottom line gives us $y+2z=10$. Quick inspection gives $(2,4),(4,3),(6,2)$ and $(8,1)$ as the only positive solutions to this equation. Plugging these solutions back into $x+y+z=31$ gives the solutions $(22,8,1),(23,6,2),(24,4,3)$ and $(25,2,4)$ as the only positive solutions to this system of equations. □

S3.6) Find the five different ways a collection of 100 coins - pennies, dimes, and quarters - can be worth exactly \$4.99.

Proof. Subtracting the first equation from the second, solving this difference equation and then plugging solutions back into the first equations gives us the five solutions:

$$(59,39,2),(64,31,5),(69,23,8),(74,15,11) \text{ and } (79,7,14).$$

$\square$

S4.2) Find the least residue of 1789 (mod 4), (mod 10), and (mod 101).

Proof. Using the division algorithm we find that $1789 \equiv 1 \pmod 4$, $1789 \equiv 9 \pmod{10}$ and $1789 \equiv 72 \pmod{101}$. $\square$

S4.3) Prove or disprove that if $a \equiv b \pmod m$, then $a^2 \equiv b^2 \pmod m$.

Proof. By definition $m \mid a - b$. Thus $m \mid (a+b)(a-b) = a^2 - b^2$. We conclude that $a^2 \equiv b^2 \pmod m$. $\square$

S4.4) Prove or disprove that if $a^2 \equiv b^2 \pmod m$, then $a \equiv b$ or $-b \pmod m$.

Proof. Note that $10^2 \equiv 3^2 \pmod{91}$ since $7 \cdot 13 = 91 \mid (10-3)(10+3) = 7 \cdot 13$. However, 91 does not divide neither 7 nor 13. $\square$

S4.6) Find all m such that $1848 \equiv 1914 \pmod m$.

Proof. This is equivalent to finding the m such that $m \mid 1914 - 1848 = 66$. $66 = 2 \cdot 3 \cdot 11$ and so $m \in \{2, 3, 11, 6, 22, 33, 66\}$. $\square$

A1) Let $\tau(n)$ be the number of positive divisors of n. Show that $\tau(n) = \tau(n+1) = \tau(n+2) = \tau(n+3)$ if $n = 3655$.

Proof. Note that

$$3655 = 5 \cdot 17 \cdot 43$$
$$3656 = 2^3 \cdot 457$$
$$3657 = 3 \cdot 23 \cdot 53$$
$$3658 = 2 \cdot 31 \cdot 59$$
$$\tau(3655) = 2 \cdot 2 \cdot 2 = 8$$
$$\tau(3656) = 4 \cdot 2 = 8$$
$$\tau(3657) = 2 \cdot 2 \cdot 2 = 8$$
$$\tau(3658) = 2 \cdot 2 \cdot 2 = 8.$$

$\square$

A2) Prove that the integer $53^{103} + 103^{53}$ is divisible by 39, and that $111^{333} + 333^{111}$ is divisible by 7.

Proof. Note that

$$\begin{aligned} 53^{103} + 103^{53} &\equiv 14^{103} + (-14)^{53} \pmod{39} \\ &\equiv \left(14^2\right)^{51} \cdot 14 + \left((-14)^2\right)^{26} \cdot (-14) \pmod{39} \\ &\equiv 14 + (-14) \pmod{39} \\ &\equiv 0 \pmod{39} \end{aligned}$$

and that

$$\begin{aligned} 111^{333} + 333^{111} &\equiv 1^{333} + (-3)^{111} \pmod{7} \\ &\equiv 1 + \left((-3)^3\right)^{37} \pmod{7} \\ &\equiv 1 + (-1) \pmod{7} \\ &\equiv 0 \pmod{7} \end{aligned}$$

□

A3) For each $n \geq 1$, use congruence theory to establish each of the following statements.

a) $7 \mid 5^{2n} + 3 \cdot 2^{5n-2}$

Proof. Note that

$$\begin{aligned} 5^{2n} + 3 \cdot 2^{5n-2} &\equiv 25^n + 24 \cdot 2^{5(n-1)} \pmod{7} \\ &\equiv (-3)^n + 3 \cdot 32^{n-1} \pmod{7} \\ &\equiv (-3)^n + 3 \cdot (-3)^{n-1} \pmod{7} \\ &\equiv 0 \pmod{7} \end{aligned}$$

□

b) $13 \mid 3^{n+2} + 4^{2n+1}$

Proof. Note that

$$\begin{aligned} 13^{n+2} + 4^{n2+1} &\equiv 9 \cdot 3^n + 4 \cdot 16^n \pmod{13} \\ &\equiv (-4) \cdot 3^n + 4 \cdot 3^n \pmod{13} \\ &\equiv 0 \pmod{13} \end{aligned}$$

□

c) $27 \mid 2^{5n+1} + 5^{n+2}$.

Proof. Note that

$$\begin{aligned} 2^{5n+1} + 5^{n+2} &\equiv 2 \cdot 32^n + 25 \cdot 5^n \pmod{27} \\ &\equiv 2 \cdot 5^n + (-2) \cdot 5^n \pmod{27} \\ &\equiv 0 \pmod{27} \end{aligned}$$

□

A4) Let a be an odd integer. Use induction on n to prove that for any $n \geq 1$,

$$a^{2^n} \equiv 1 \pmod{2^{n+2}}.$$

Proof. (Base case, $n = 1$) Clearly $a^{2^1} = a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1 \equiv 1 \pmod 4$.

(Inductive step) For our inductive hypothesis, assume that $a^{2^k} \equiv 1 \pmod{2^{k+2}}$ for some integer k. We want to prove the statement for $n = k+1$. Note that

$$\begin{aligned} a^{2^{k+1}} &= \left(a^{2^k}\right)^2 \\ &= \left(2^{k+2}k' + 1\right)^2 \\ &= 2^{2k+4}\left(k'\right)^2 + 2^{k+3}k' + 1 \\ &= 1 \pmod{2^{k+3}} \end{aligned}$$

since $2k + 4 > k + 3$ for $k \geq 1$. □

A5) Prove that, for any natural numbers m and n, the number $3^m + 3^n + 1$ is never a perfect square. [Hint: Work modulo 8].

Proof. Some calculation shows that the residue set of powers of 3 (mod 8) are $\{1, 3\}$. Thus the residue set of numbers of the form $3^m + 3^n + 1$ is $\{3, 5, 7\}$. The residue set of squares (mod 8) is $\{0, 1, 4\}$. Since these sets are disjoint, we conclude that $3^m + 3^n + 1$ could never be a perfect square. □

Extra Credit Problems.

EC1) A certain prison row has 1000 cells, each holding one prisoner. A jailer, carrying out the terms of a partial amnesty, unlocked every cell in the prison row. Next he locked every second cell. Then he turned the key in every third cell, locking those cells which were open and opening those cells which were locked. He continued in this way, on the n-th trip turning the key in every n-th cell, for $n = 1, 2, \ldots, 1000$. At the end of the process, those prisoners whose cells remained open were allowed to go free. How many prisoners were set free? Explain your reasoning.

Proof. Note that for each divisor of n the key in cell n is turned exactly once (and it isn't turned at any other time). Thus it is turned $\tau(n)$ times. The prisoner in cell n is set free if the key is turned an odd number of times so we require that $\tau(n) \equiv 1 \pmod 2$. Suppose $n = p_1^{k_1} \cdots p_r^{k_r}$. Then $\tau(n) = (k_1 + 1) \cdots (k_r + 1)$ is odd iff each factor $(k_i + 1)$ is odd iff each k_i is even iff n is a perfect square. The first perfect square in this interval is 1 and the last perfect square in this interval is $961 = 31^2$ and so we conclude that exactly $31 - 1 + 1 = 31$ prisoners were set free. □

EC2) A positive integer is called *polite* if it can be represented as a sum of two or more consecutive natural numbers. For example, 7 and 22 are polite since $7 = 3 + 4$ and $22 = 4 + 5 + 6 + 7$, while 2 is impolite. Prove that the only impolite positive integers are the powers of 2, that is, 1, 2, 4, 8, 16, ...

Proof. See `http://en.wikipedia.org/wiki/Polite_number`, an article about polite numbers. □