Math 406 – Fall 2009 – Harry Tamvakis

PROBLEM SET 5 – Due October 22, 2009

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Problems

7.12 For which n is $\sigma(n)$ odd?

Proof. Let $n = 2^{k_0} p_1^{k_1} \cdots p_r^{k_r}$. Note that

$$\sigma(n) = \left(1 + 2 + \cdots + 2^{k_0}\right)\left(1 + p_1 + \cdots + p_1^{k_1}\right)\cdots\left(1 + p_r + \cdots + p_r^{k_r}\right).$$

In order for $\sigma(n)$ to be odd, we require that each individual factor above is odd. Note that it doesn't matter what k_0 is because $1 + 2 + \cdots + 2^{k_0}$ will always be odd. Also, $1 + p_i + \cdots + p_i^{k_i}$ will be odd iff k_i is even (since p_i is odd for $i > 0$). Thus $\sigma(n)$ is odd iff $k_1, \ldots, k_r \equiv 0 \pmod 2$. □

7.16 Find infinitely many n such that $\sigma(n) \leq \sigma(n-1)$.

Proof. Consider the set $P = \{p \mid p \text{ prime and } p-1 \text{ composite}\}$. Clearly P is infinite since the only primes not in P are 2 and 3. Let $p \in P$ be arbitrary. Note that $\sigma(p) = p+1$. However, $p-1$ is composite so among its divisors are $1, p-1$ and d where $1 < d < p-1$. Thus $\sigma(p-1) \geq 1+p-1+d = p+d$. Since $d > 1$, we have that $\sigma(p-1) \geq p+d \geq p+1 = \sigma(p)$. Thus $\sigma(p-1) \geq \sigma(p)$. □

8.2 It was long thought that even perfect numbers ended alternately in 6 and 8. Show that this is wrong by verifying that the perfect numbers corresponding to the primes $2^{13}-1$ and $2^{17}-1$ both end in 6.

Proof. Note that

$$\begin{aligned} 2^{12}\left(2^{13}-1\right) &\equiv 4096 \cdot 1891 \\ &\equiv 6 \cdot 1 \\ &\equiv 6 \pmod{10} \\ 2^{16}\left(2^{17}-1\right) &\equiv 65536 \cdot 131071 \\ &\equiv 6 \cdot 1 \\ &\equiv 6 \pmod{10}. \end{aligned}$$

□

8.5 If $\sigma(n) = kn$, then n is called a k-perfect number. Verify that 672 is 3−perfect and $2,178,540 = 2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19$ is 4−perfect.

Proof.

$$\begin{aligned}
\sigma(672) &= \sigma\left(2^5 \cdot 3 \cdot 7\right) \\
&= \left(\frac{2^6-1}{2-1}\right)(1+3)(1+7) \\
&= 63 \cdot 4 \cdot 8 \\
&= 2016 \\
&= 3 \cdot 672 \\
\sigma(2178540) &= \sigma\left(2^2 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13 \cdot 19\right) \\
&= \left(\frac{2^3-1}{2-1}\right)\left(\frac{3^3-1}{3-1}\right)(1+5)\left(\frac{7^3-1}{7-1}\right)(1+13)(1+19) \\
&= 7 \cdot 13 \cdot 6 \cdot 57 \cdot 14 \cdot 20 \\
&= 8714160 \\
&= 4 \cdot 2178540.
\end{aligned}$$

□

8.7 Let us say that n is superperfect if and only if $\sigma(\sigma(n)) = 2n$. Show that if $n = 2^k$ and $2^{k+1} - 1$ is prime, then n is superperfect.

Proof.

$$\begin{aligned}
\sigma(\sigma(n)) &= \sigma\left(\sigma\left(2^k\right)\right) \\
&= \sigma\left(\frac{2^{k+1}-1}{2-1}\right) \\
&= \sigma\left(2^{k+1}-1\right) \\
&= 1 + \left(2^{k+1}-1\right) \\
&= 2^{k+1} \\
&= 2 \cdot 2^k \\
&= 2n.
\end{aligned}$$

□

8.13 Show that all even perfect numbers end in 6 or 8.

Proof. Note that (mod 10) we have $2^{k_1} \equiv 2, 2^{k_2} \equiv 4, 2^{k_3} \equiv 8, 2^{k_4} \equiv 6, 2^{k_5} \equiv 2$ where $k_i > 0$ and $k_i \equiv i \pmod{\phi(10) = 4}$. Also note that every even perfect number is of the form $2^{p-1}(2^p - 1)$ for some prime p. Note that when $p = 2$, $2^{2-1}(2^2 - 1) = 6$, so without loss of generality, let $p \geq 3$, in particular, p is odd. Thus $p \equiv 1$ or $3 \pmod 4$. Now we have two cases. When $p \equiv 1 \pmod 4$, then $2^{p-1}(2^p - 1) \equiv 6(2-1) \equiv 6 \pmod{10}$. When $p \equiv 3 \pmod 4$ then $2^{p-1}(2^p - 1) \equiv 4 \cdot (8-1) \equiv 8 \pmod{10}$. □

9.2 Calculate $\phi(54), \phi(540)$, and $\phi(5400)$.

Proof.

$$\begin{aligned}
54 &= 2 \cdot 3^3 \\
\phi(54) &= (2-1)\,3^2\,(3-1) \\
&= 2 \cdot 3^2 \\
&= 18 \\
540 &= 2^2 \cdot 3^3 \cdot 5 \\
\phi(540) &= 2\,(2-1)\,3^2\,(3-1)\,(5-1) \\
&= 2^4 \cdot 3^2 \\
&= 144 \\
5400 &= 2^3 \cdot 3^3 \cdot 5^2 \\
\phi(5400) &= 2^2\,(2-1)\,3^2\,(3-1)\,5\,(5-1) \\
&= 2^5 \cdot 3^2 \cdot 5 \\
&= 1440.
\end{aligned}$$

□

9.14 Find four solutions of $\phi(n) = 16$.

Proof. Some solutions are $n = 17, 32, 34, 40, 48, 60$. □

9.18 Show that $\phi(n) = n/2$ if and only if $n = 2^k$ for some positive integer k.

Proof. Suppose $n = 2^k$. Then $\phi(n) = \phi\left(2^k\right) = 2^{k-1}(2-1) = 2^{k-1} = 2^k/2$.

Next suppose that $\phi(n) = n/2$. This implies that n is even and in particular, is of the form $n = 2^k m$ where m is odd. Note that $2^{k-1}m =$

$\frac{n}{2} = \phi(n) = \phi(2^k)\phi(m) = 2^{k-1}\phi(m)$. This implies that $m = \phi(m)$, in particular, $m = 1$. $\square$

A1) Prove that n is a perfect number if and only if $\displaystyle\sum_{d|n} \frac{1}{d} = 2$. For example, 6 is perfect because

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2.$$

Proof. Clearly $\sum_{d|n} \frac{1}{d} = 2$ iff $n\sum_{d|n} \frac{n}{d} = \sum_{d|n} \frac{n}{d} = 2n$. Note that as d runs through the divisors of n, $\frac{n}{d}$ also runs through the divisors of n. Thus $\sum_{d|n} \frac{d}{n} = \sum_{d|n} d = \sigma(n)$ and so $\sum_{d|n} \frac{n}{d} = 2n$ iff $\sigma(n) = 2n$ iff n is perfect. $\square$

A2) Find the last two digits in the decimal expansions of 3^{1000} and 7^{999999}.

Proof. Note that $\phi(100) = \phi(2^2 \cdot 5^2) = 2(2-1)5(5-1) = 40$. Furthermore, $1000 \equiv 0 \pmod{40}$ and $999999 \equiv 39 \pmod{40}$. Thus $3^{1000} \equiv 3^0 \equiv 1 \pmod{100}$ and $7^{999999} \equiv 7^{39} \equiv 43 \pmod{100}$. $\square$

A3) Prove that $2^{15} - 2^3$ divides $a^{15} - a^3$ for any integer a. [Hint: $2^{15} - 2^3 = 5 \cdot 7 \cdot 8 \cdot 9 \cdot 13$.]

Proof. Note that $\phi(5) = 4, \phi(7) = 6, \phi(8) = 4, \phi(9) = 6$ and $\phi(13) = 12$. Furthermore, $15 \equiv 3 \pmod{4, 6, 12}$. By Euler's Theorem we have that $4, 5, 8, 9, 13 \mid a^{15} - a^3$ for any integer a. Thus $2^{15} - 2^3 \mid a^{15} - a^3$. $\square$

<u>Extra Credit Problems.</u>

EC1) (a) Consider the number $m = 111\cdots1$ with n digits, all ones. Prove that if m is prime, then n is prime.

Proof. Note that $m = \frac{1}{9}(10^n - 1)$. Suppose $n = ab$ is not prime $(a, b > 1)$. Then $m = \frac{1}{9}(10^a - 1)\left(1 + 10^a + 10^{2a} + \cdots + 10^{a(b-1)}\right)$. Clearly $\frac{1}{9}(10^a - 1)$ is an integer greater than 1 because $a > 1$. Furthermore, $\left(1 + 10^a + 10^{2a} + \cdots + 10^{a(b-1)}\right)$ is also an integer greater than 1. We conclude that m is composite. $\square$

(b) Is the converse of the statement in (a) true?

Proof. No, note that $n = 5$ is prime but $m = 11111 = 41 \cdot 271$ is not. $\square$

EC2) Consider the canonical factorization of 1000! into prime powers:

$$1000! = 2^a 3^b 5^c 7^d 11^e \cdots$$

Compute the exponents a, b, c, and d.

Proof. Counting the factors of $2, 3, 5$ and 7 in 1000! gives us $a = 994, b = 498, c = 249$ and $d = 164$. $\square$