Math 406, Introduction to Number Theory 10/8/09

SOLUTIONS TO MIDTERM 1

1) We can divide out by $(40, 64) = 8$, and solve the equivalent equation $5x + 8y = 7$. One solution (x_0, y_0) is easily found, e.g. $x_0 = 3$, $y_0 = -1$. Now all solutions are given by $x = 3 + 8t$, $y = -1 - 5t$, where $t \in \mathbb{Z}$.

2) $1453 \equiv 2713 \pmod{n}$ means that n divides $2713 - 1453 = 1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$. The only squares $n > 1$ that divide 1260 are 4, 9, and 36.

3) Since $5 \mid n$ we must have $y = 0$ or $y = 5$, and since $3 \mid n$ we have $2 + 3 + x + 4 + 5 + 6 + 7 + 8 + y \equiv 0 \pmod{3}$, or $x + y \equiv 1 \pmod{3}$. If $y = 0$, we get $x \in \{1, 4, 7\}$, while if $y = 5$, we get $x \in \{2, 5, 8\}$. This gives the six possible numbers

$$231456780,\ 234456780,\ 237456780,\ 232456785,\ 235456785,\ 238456785.$$

4) We have

$$\begin{aligned} 77777777 &= 1000 \cdot 77777 + 777 \\ 77777 &= 100 \cdot 777 + 77 \\ 777 &= 10 * 77 + 7 \\ 77 &= 11 * 7 + 0. \end{aligned}$$

We deduce that $\gcd(77777777, 77777) = 7$.

5) Note that $210 = 2 \cdot 3 \cdot 5 \cdot 7$. One possible answer is $n = 2 \cdot 3^2 \cdot 5^4 \cdot 7^6$.

6) Suppose that $d \geq 1$ divides both a and c. Since $c \mid a + b$, we see that $d \mid a + b$. Since $d \mid a$ and $d \mid a + b$, we get $d \mid (a + b) - a$, i.e., $d \mid b$. But then $d = 1$, because $(a, b) = 1$. This proves that $(a, c) = 1$.

7) We have $1835 \equiv 1 \pmod{7}$, hence $1835^{1910} \equiv 1 \pmod{7}$. Also, $1986 \equiv 5 \pmod{7}$, so $1986^{2061} \equiv 5^{2061} \pmod{7}$. By Fermat's theorem, we have $5^6 \equiv 1 \pmod{7}$, and since $2061 = 6 \cdot 343 + 3$, we obtain $5^{2061} \equiv 5^3 \equiv 6 \pmod{7}$. Finally, we deduce that

$$1835^{1910} + 1986^{2061} \equiv 1 + 6 = 0 \pmod{7}.$$

8) It suffices to show that both 5 and 7 divide $n = ab(a^{12} - b^{12})$. If $5 \mid a$ or $5 \mid b$, then $5 \mid n$, so assume that $(a, 5) = (b, 5) = 1$. By Fermat's theorem, we then have $a^4 \equiv 1 \pmod{5}$, therefore $a^{12} \equiv 1 \pmod{5}$. Similarly, we see that $b^{12} \equiv 1 \pmod{5}$, hence $5 \mid a^{12} - b^{12}$, and so $5 \mid n$. The exact same argument, with 7 in place of 5 and 6 in place of 4, shows that $7 \mid n$. We deduce that $35 = 5 \cdot 7 \mid n$.

EC) Let n be a number which is not a square, and suppose that n has τ positive divisors. To each divisor d of n we may associate the divisor n/d, with the property that $d \cdot (n/d) = n$. Note that we never have $d = n/d$ because $n \neq d^2$ by assumption. In this way, the τ divisors of n are partitioned into $\tau/2$ pairs, with the product of the two divisors in each pair equal to n. We deduce that the product of all the positive divisors of n is equal to $n^{\tau/2}$. Now since $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$ has $(8+1)(4+1)(2+1)(1+1) = 270$ divisors, we see that their product is equal to $(10!)^{135}$.