

## The Quotient vector Space

Suppose  $V$  is a vector space over  $K$  and  $U \subset V$  is a subspace. We will describe a construction of the quotient vector space  $V/U$ .

But first we will discuss equivalence relations. If  $S$  is a set then a relation  $\sim$  on  $S$  is some way of relating elements of  $S$ . The expression  $x \sim y$  means  $x$  is related to  $y$ . For example if  $S$  is all the people in the world, then  $x \sim y$  might mean ' $x = y$  or  $x$  is a brother or sister of  $y$ ' and  $x \approx y$  might mean ' $x$  is a mother of  $y$ '. Both  $\sim$  and  $\approx$  are relations.

We say a relation  $\sim$  is an equivalence relation if

- a)  $x \sim x$  for all  $x \in S$ .
- b) If  $x \sim y$  then  $y \sim x$ .
- c) If  $x \sim y$  and  $y \sim z$  then  $x \sim z$ .

Thus in the examples above  $\sim$  is an equivalence relation but  $\approx$  is not. For a mathematical example, let  $S$  be the integers and say that  $x \sim y$  if and only if  $x - y$  is even. This is an equivalence relation.

If you have an equivalence relation  $\sim$  then  $S$  can be divided up into equivalence classes. An equivalence class is a set of the form  $[x] = \{y \in S \mid y \sim x\}$ . Note that if  $x \sim y$  then  $[x] = [y]$ . So two different equivalence classes are always disjoint. In the integer example above there are just two equivalence classes, the even integers and the odd integers. We let  $S/\sim$  denote the set of equivalence classes of  $S$ . We say that  $x$  is a representative of the equivalence class  $[x]$ . If  $y \sim x$  then  $y$  is also a representative of  $[x]$ .

Now let us go back to vector spaces. Suppose  $V$  is a vector space over  $K$  and  $U \subset V$  is a subspace. We define an equivalence relation  $\sim$  on  $V$  by  $x \sim y$  iff  $x - y \in U$ . Let  $V/U = V/\sim$ . Define addition and scalar multiplication on  $V/U$  by

- a)  $[x] + [y] = [x + y]$
- b)  $c[x] = [cx]$

We must show these operations do not depend on which representative  $x$  we choose. So suppose  $[x] = [x']$  and  $[y] = [y']$ . We want to show that  $[x + y] = [x' + y']$ , i.e., that  $x + y \sim x' + y'$ . But  $(x + y) - (x' + y') = x - x' + y - y'$  and  $x - x' \in U$  and  $y - y' \in U$ . So  $(x + y) - (x' + y') \in U$  which means  $x + y \sim x' + y'$  which means  $[x + y] = [x' + y']$ . Likewise  $(cx) - (cx') = c(x - x') \in U$  so  $[cx] = [cx']$ . You can show all the vector space axioms are satisfied, so  $V/U$  is a vector space over  $K$ .

For example, let  $V = \mathbb{C}^2$  and  $U = \{(x, y) \mid x = 2y\}$ . Then  $(x, y) \sim (x', y')$  iff  $x - x' = 2(y - y')$  iff  $x - 2y = x' - 2y'$ . Note that if  $W$  is a complementary subspace to  $U$  (i.e.,  $U \oplus W = V$ ) then for each  $[(x, y)] \in V/U$  there is a unique representative  $(x', y')$  in  $W$ . So we may identify  $V/U$  with  $W$ . For example suppose  $W$  is the  $x$  axis. Then  $(x, y) \sim (x - 2y, 0)$  on the  $x$  axis.

For another example, let  $V = \mathbb{R}^{\mathbb{R}}$  and let  $U = \{f \in V \mid f(t) = 0 \text{ for all } t \in [0, 1]\}$ . Then two functions are equivalent if they agree on  $[0, 1]$ . Thus  $V/U$  can be identified with  $\mathbb{R}^{[0, 1]}$ , the functions from  $[0, 1]$  to  $\mathbb{R}$ .